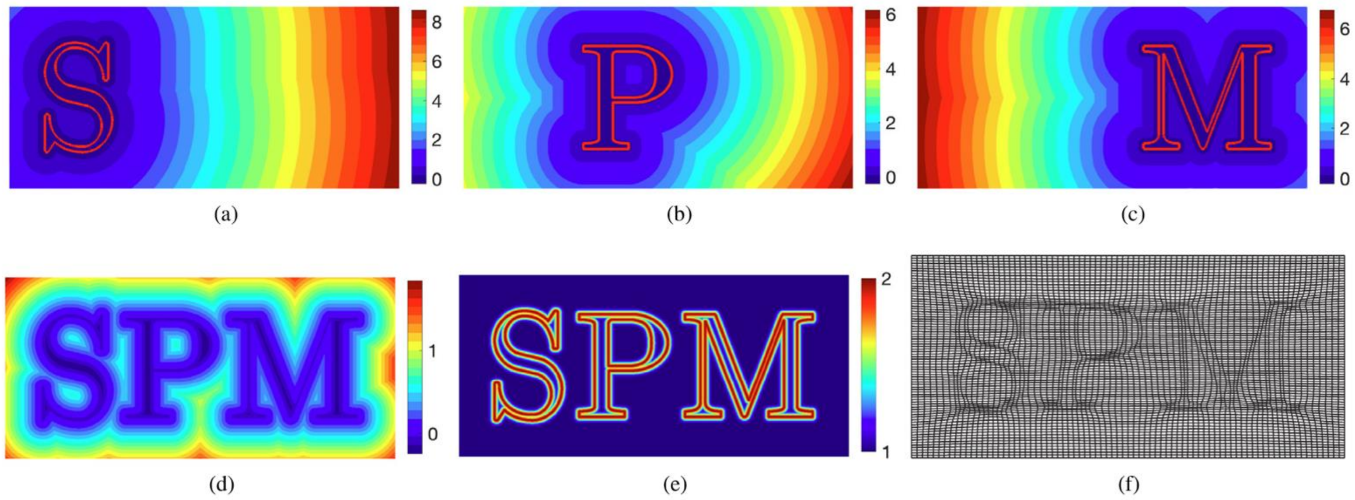

Extended r-Adaptive Isogeometric Analysis for Weak-Discontinuous Problems

Resolving gradient jumps by coupling partition-of-unity enrichment with level-set-driven control-point relocation

SYMPOSIUM ON SOLID & PHYSICAL MODELING · SPM 2026

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Outline

1. Background and motivation
2. Method and algorithms
3. Benchmarks and results
4. Conclusions and outlook

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Weak discontinuities at material interfaces

- Composites, layered structures, porous solids, heterogeneous diffusion — **interfaces are ubiquitous** in engineering and biomedicine.
- The primary field stays **continuous** across the interface, but its **gradient jumps** because material coefficients are discontinuous.
- These **weak discontinuities** arise in elasticity with inclusions, multi-phase heat conduction, and transport in stratified media.
- Accuracy degrades sharply unless the discretization is **adapted to the interface geometry**.

Weak discontinuity: the solution is C^0 but not C^1 across Γ — a kink, not a cliff.

High-order smoothness fights the kink; the remedy is geometry, not more smoothness.

Pronounced material interfaces: (top-left) steel reinforcement in a cementitious matrix; (top-right, bottom-left) multi-material LPBF metal 3D printing with internal material interfaces; (bottom-right) a dental implant in cortical/trabecular bone — all with strong stiffness contrast.

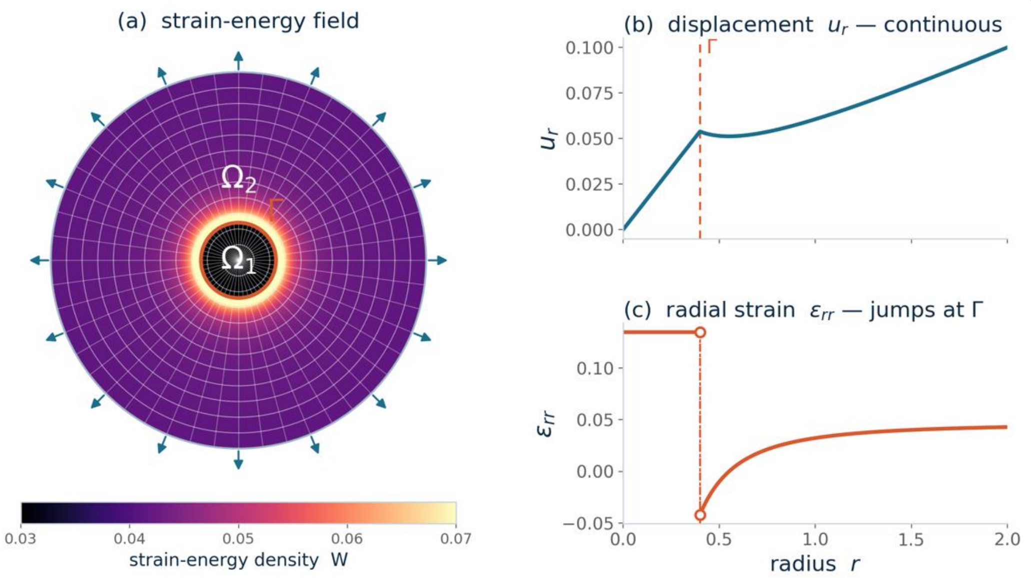
Interface requirements and the limits of existing tools

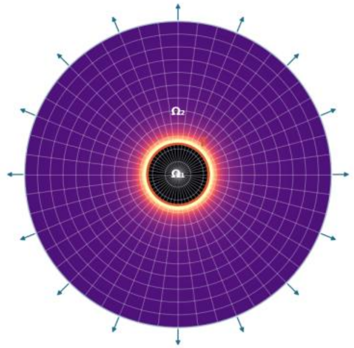
At every interface Γ_I^k

$$[\![\mathbf{u}]\!] = \mathbf{0}, \quad [\![\boldsymbol{\sigma}\mathbf{n}_I^k]\!] = \mathbf{0} \quad \text{on } \Gamma_I^k$$

displacement + traction continuity across dissimilar materials
⇒ the gradient jumps.

- **Interface-fitted meshes / strong h-refinement** — accurate but **cumbersome for complex geometry**.
- **XFEM enrichment** — captures the jump, but **adds DOF** and needs special cut-element quadrature.
- **Standard IGA** — exact CAD & high smoothness, but that smoothness is **wrong across a kink**.
- **r-adaptivity alone** — relocates points, preserves CAD, yet **cannot reproduce the derivative jump**.





The signature

Displacement (b) stays continuous across Γ; the radial strain (c) jumps — a kink, not a gap. Standard high-order smoothness cannot capture it.

Existing approaches and their limitations

Method	Captures gradient jump	Exact CAD preserved	Adds DOF	Key limitation
Interface-fitted FEM / h-refinement	✓	✗	—	Re-meshing is cumbersome for complex geometry
XFEM enrichment	✓	✗	✗	Extra DOF + special cut-element quadrature
Standard IGA	✗	✓	✗	Smoothness is wrong across a kink
XIGA (enriched IGA)	✓	✓	✗	Geometric resolution still misaligned with Γ
r-adaptivity (alone)	✗	✓	—	Cannot reproduce the derivative jump
This work: XIGA + r-adaptivity	✓	✓	**fixed**	***Combines both — < 1% overhead***

The dominant error stems from insufficient geometric resolution of the interface, rather than from a lack of degrees of freedom.

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Coupling enrichment with control-point relocation

Enrichment provides the approximation space needed to represent the **gradient jump**; **r-adaptivity** redistributes the existing resolution toward the interface. CAD geometry, spline topology and the degree-of-freedom count are **all preserved**.

Coupled formulation

Enrichment and r-adaptivity **unified within a single isogeometric framework** for weak discontinuities.

Geometry-aware monitor

A Gaussian monitor built from a KS-aggregated level set redistributes resolution **by interface geometry, not by refinement**.

CAD-consistent discretization

Exact geometry, spline topology and global continuity preserved; the **number of basis functions is unchanged**.

Validated accuracy and efficiency

Improves on standard IGA, enrichment-only XIGA and h-refined schemes **across four benchmarks**.

Headline result: up to **65.7% lower energy-norm error** at fixed DOF, with the relocation step costing **under 1%** of the XIGA solve.

Linear elasticity with weak discontinuities

Strong form

Equilibrium on Ω

$$\nabla\cdot\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } \Omega$$

Interface conditions

continuity of u and traction

$$[\![\mathbf{u}]\!] = \mathbf{0}, \quad [\![\boldsymbol{\sigma}\mathbf{n}_l^k]\!] = \mathbf{0} \quad \text{on } \Gamma_l^k$$

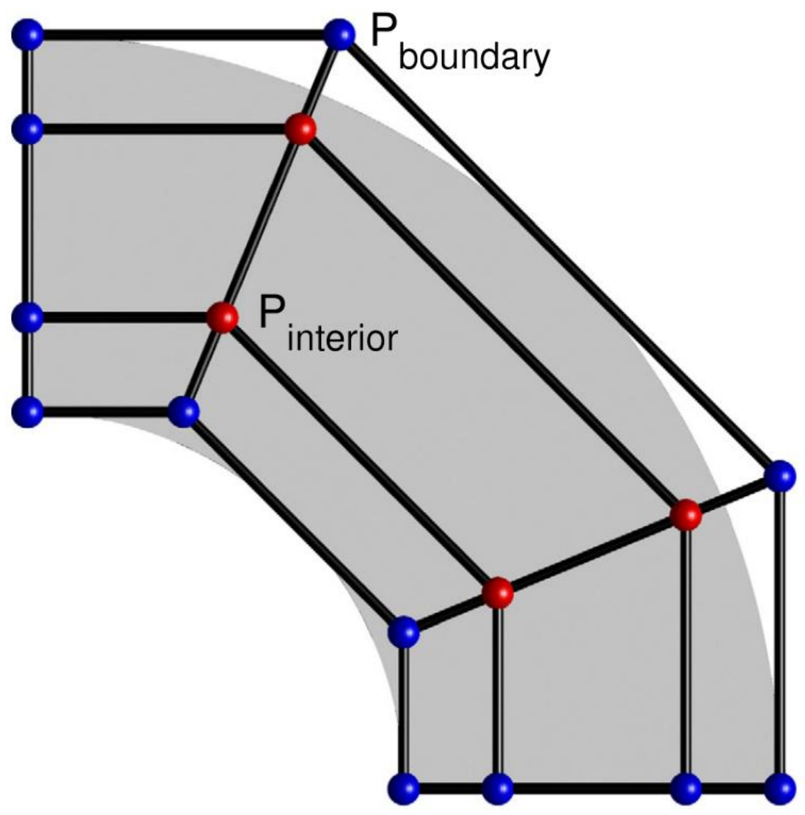
Weak form

find u ∈ $\mathcal{U}$ (energy minimiser)

$$a(\mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathcal{V}$$

- Hooke's law $\sigma = \mathbb{D} : \varepsilon$ with $\mathbb{D}$ symmetric positive-definite and **piecewise-constant on each subdomain**.
- Material parameters are discontinuous across Γ_l^k — hence the **displacement gradient may jump** while u itself stays continuous.
- This variational interface problem is the **model setting for the extended r-adaptive scheme** that follows.

Control points are geometric degrees of freedom



Interior control points are design variables; boundary points stay fixed. Relocating them changes the parameterization while preserving topology and bijectivity.

$$x(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,p}(\xi)\, M_{j,q}(\eta)\, \mathbf{P}_{ij}$$

Tensor-product B-spline / NURBS map $x : \hat{\Omega} \to \Omega$, bijective with $\det(\nabla x) > 0$.

- Splines parameterize the geometry and approximate the solution — **exact CAD, high-order smoothness**.
- Split the control net: fixed boundary set $\mathcal{I}_{\mathrm{B}}$ + **free interior set $\mathcal{I}_{\mathrm{I}}$**.
- Interior points carry **geometric freedom** — usable for mapping quality, or for anisotropic, solution-aware distributions.
- **This is the lever for r-adaptivity: move interior points, keep the spline space.**

Partition-of-unity enrichment for the gradient jump

Enriched approximation

$$\mathbf{u}_h(\boldsymbol{\xi}) = \sum_{i \in \mathcal{N}_{\text{std}}} R_i(\boldsymbol{\xi})\,\mathbf{u}_i + \sum_{j \in \mathcal{N}_{\text{faces}}} R_j(\boldsymbol{\xi})\,\Psi(\boldsymbol{\xi})\,\mathbf{u}_j$$

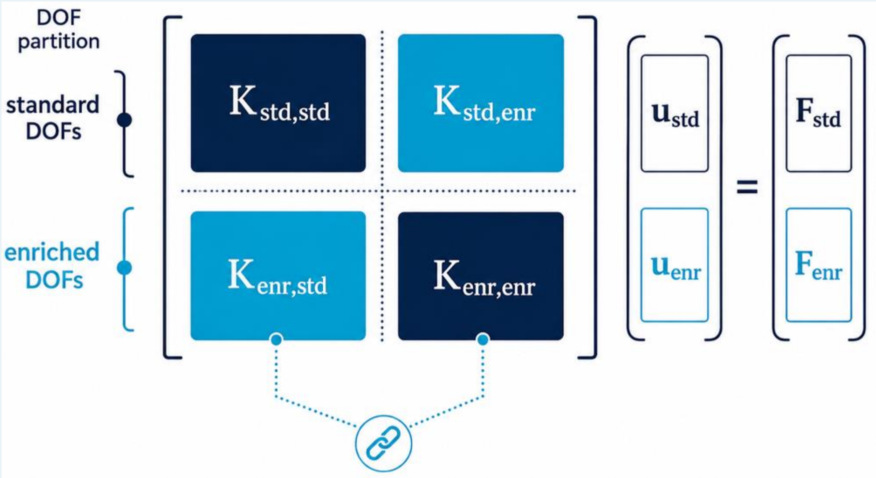
Enrichment function (shift–subtraction, abs-value type)

$$\Psi(\boldsymbol{\xi}) = \sum_I |\varphi_I|\, R_I(\boldsymbol{\xi}) - \left|\sum_I \varphi_I R_I(\boldsymbol{\xi})\right|$$

- Only control points whose **support cuts Γ** are enriched (set $\mathcal{N}_{\text{faces}}$).
- The shift–subtraction form keeps the **partition-of-unity property** intact.
- **Result is C^0 but not C^1 across Γ — exactly the regularity of a weak discontinuity.**
- Interface represented implicitly by the **zero level-set** of a signed-distance $\varphi^{(k)}(x)$.

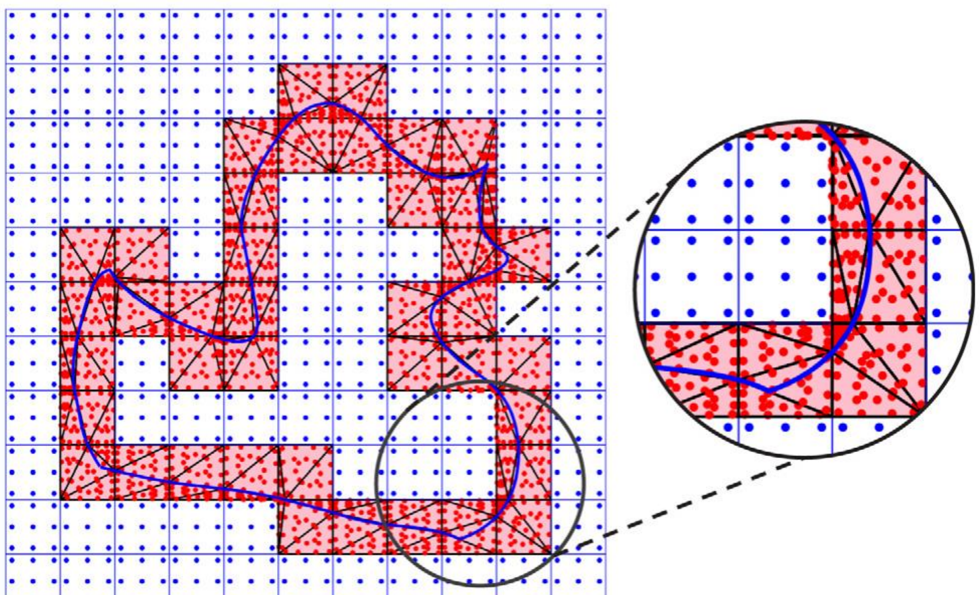
Coupled standard-enriched system

Discrete system in block form:



Off-diagonal blocks encode coupling between spline and enrichment DOFs.

Sub-triangulated quadrature and a recovery-based estimator



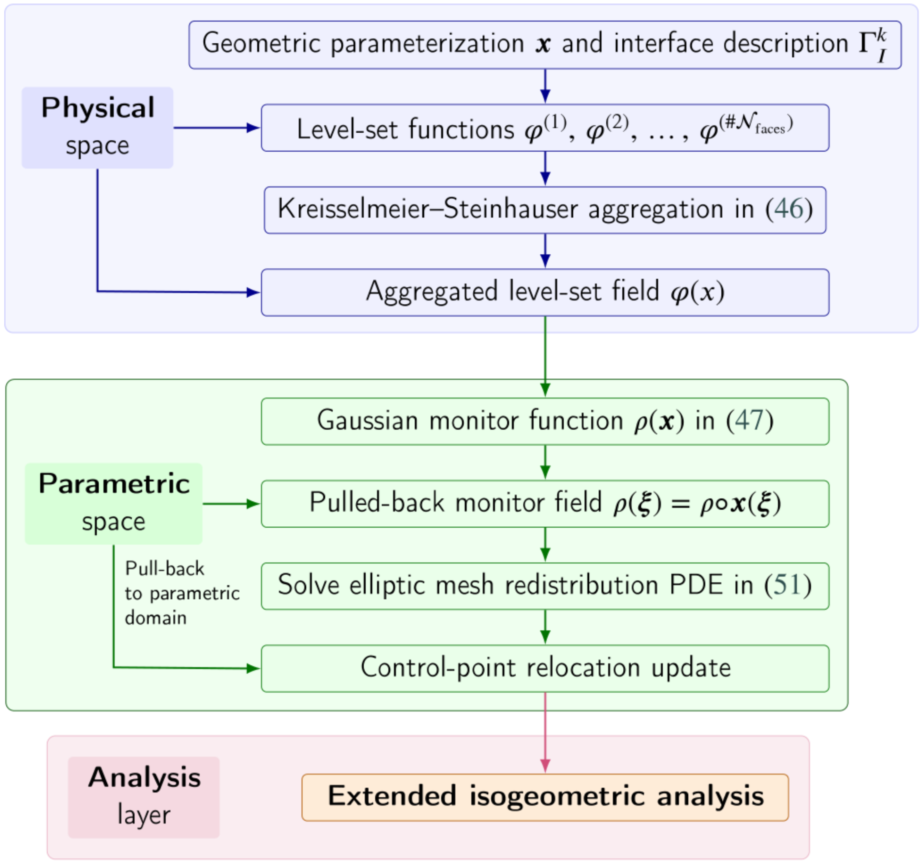
Elements cut by the inclusion are split into interface-conforming subtriangles; a 13-point Gauss rule integrates the enriched, piecewise-smooth integrands.

- Standard elements: (p+1)×(q+1) **tensor-product Gauss**.
- Cut elements: **triangular subcells + 13-point rule** — accurate enriched integration without changing the spline discretization.
- Strain recovery via an **enriched Superconvergent Patch Recovery (Zienkiewicz–Zhu)** projection of the computed strain.
- **Recovery-based energy-norm error** quantifies accuracy — the metric reported in the benchmarks:

$$\eta_{r,E} = \frac{\left\| \varepsilon^s - \varepsilon^h \right\|_{E(\Omega)}}{\left\| \varepsilon^s \right\|_{E(\Omega)}} \times 100\%$$

recovery-based relative error in the energy norm

The r-adaptive XIGA framework



Physical space

Level-set functions → **KS aggregation** → a single smooth aggregated field φ(x).

Parametric space

Gaussian monitor ρ(x) is pulled back; a **weighted elliptic PDE** relocates interior control points.

Analysis layer

The r-adapted parameterization feeds the enriched XIGA solve — **same DOF, sharper interface**.

No knot insertion · no degree elevation · topology preserved

Aggregation of level sets and the Gaussian monitor

KS smooth-minimum aggregation

$$\varphi(\mathbf{x}) = -\frac{1}{\beta}\ln\left(\sum_{k=1}^{N_{\text{faces}}} \exp\bigl(-\beta\,\varphi^{(k)}(\mathbf{x})\bigr)\right)$$

differentiable distance to the nearest interface; β controls sharpness (β→∞ ⇒ pointwise min).

Gaussian interface monitor

$$\rho(\mathbf{x}) = \rho_0 + \alpha\exp\left(-\left(\frac{|\varphi(\mathbf{x})|}{\delta}\right)^2\right)$$

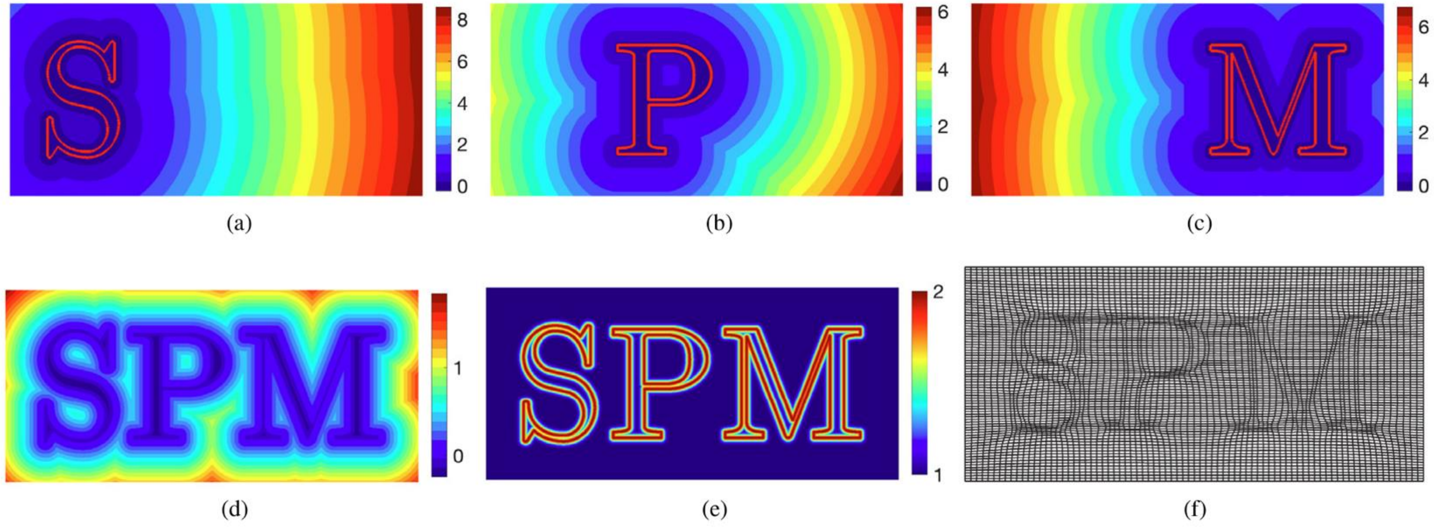
- **Peaks at $\rho_0 + \alpha$** on the interface, decays smoothly to background ρ_0.
- Smooth metric ⇒ **well-posed elliptic redistribution**; no abrupt spatial variation.
- Extends to **any number of interfaces** with no change to the discretization.

Monitor parameters

- ρ_0 — background density — resolution far from interfaces
- α — interface-driven concentration strength
- δ — width of the refinement band along Γ

contrast $\rho_{peak} / \rho_{bg} = 1 + \alpha/\rho_0$

Aggregating several interfaces into one monitor field



(a–c) individual level sets for the three letter-interfaces · (d) KS-aggregated field φ(x) · (e) Gaussian monitor ρ(x) · (f) resulting r-adapted parameterization.

- Three interfaces shaped as the letters “S”, “P”, “M” — a **fitting test for this venue**.
- **KS aggregation** fuses them into one differentiable field, with no interface-specific loops.
- Refinement zones appear **automatically around every interface**, including tight clusters.
- **Iso-parameter lines compress where ρ is large, so resolution follows the geometry.**

Control-point relocation by a linear elliptic PDE

Solve a weighted elliptic PDE

$$\nabla_{\boldsymbol{\xi}} \cdot \left(\rho(\boldsymbol{\xi}) \nabla_{\boldsymbol{\xi}} \boldsymbol{x}\right) = \mathbf{0} \qquad \text{in } \hat{\Omega}$$

Equivalently, minimizes a geometric functional

$$\mathcal{J}_{\text{geo}}(\boldsymbol{x}) = \frac{1}{2} \int_{\hat{\Omega}} \rho(\boldsymbol{\xi}) \|\nabla_{\boldsymbol{\xi}} \boldsymbol{x}\|^2 \, \mathrm{d}\boldsymbol{\xi}$$

Discretized → one SPD linear solve

$$\mathbf{K}_{\text{ff}} \mathbf{X}_{\text{f}} = -\mathbf{K}_{\text{fb}} \mathbf{X}_{\text{b}},$$

$$(\mathbf{K}_{\text{ff}})_{ij} = \int_{\hat{\Omega}} \rho(\boldsymbol{\xi}) \, \nabla_{\boldsymbol{\xi}} N_i(\boldsymbol{\xi}) \cdot \nabla_{\boldsymbol{\xi}} N_j(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi}, \;\; i, j \in \mathcal{I}_{\text{f}},$$

$$(\mathbf{K}_{\text{fb}})_{ij} = \int_{\hat{\Omega}} \rho(\boldsymbol{\xi}) \, \nabla_{\boldsymbol{\xi}} N_i(\boldsymbol{\xi}) \cdot \nabla_{\boldsymbol{\xi}} N_j(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi}, \;\; i \in \mathcal{I}_{\text{f}}, \;\; j \in \mathcal{I}_{\text{b}}.$$

Large ρ $\Rightarrow$ control points concentrate at the interfaces.

Why this matters

- Each redistribution step is a **symmetric positive-definite** linear solve.
- **No nonlinear iterations** ⇒ no dependence on an initial guess, no local minima.
- System size = control mesh size — **constant throughout the adaptive process**.
- Cost **scales linearly** with control points; the XIGA solve dominates.

Relocation cost. The relocation time stays **below 1%** of the XIGA solve time ($t_1 / t_2 < 1\%$) at every refinement level.

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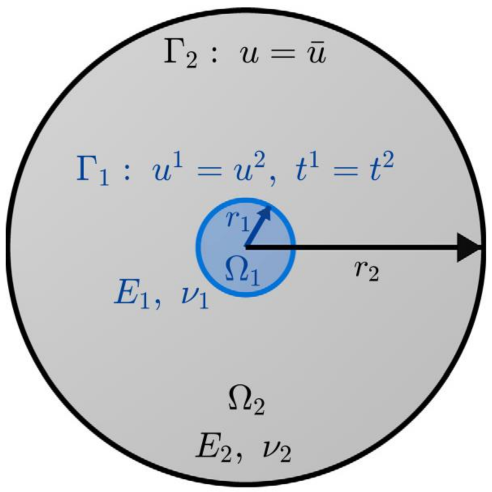
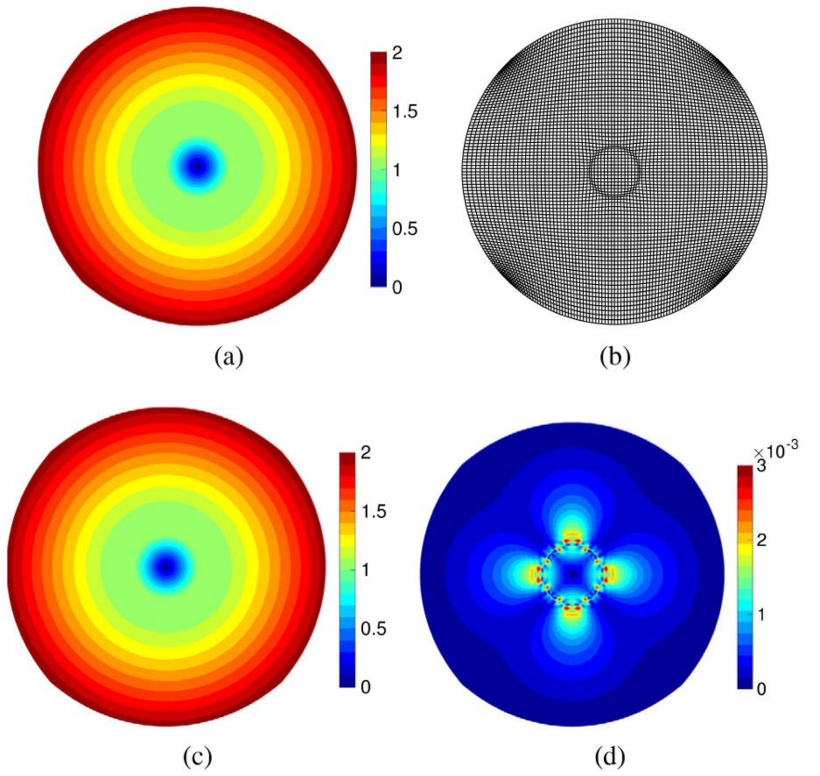
Circular bi-material inclusion with an analytical reference



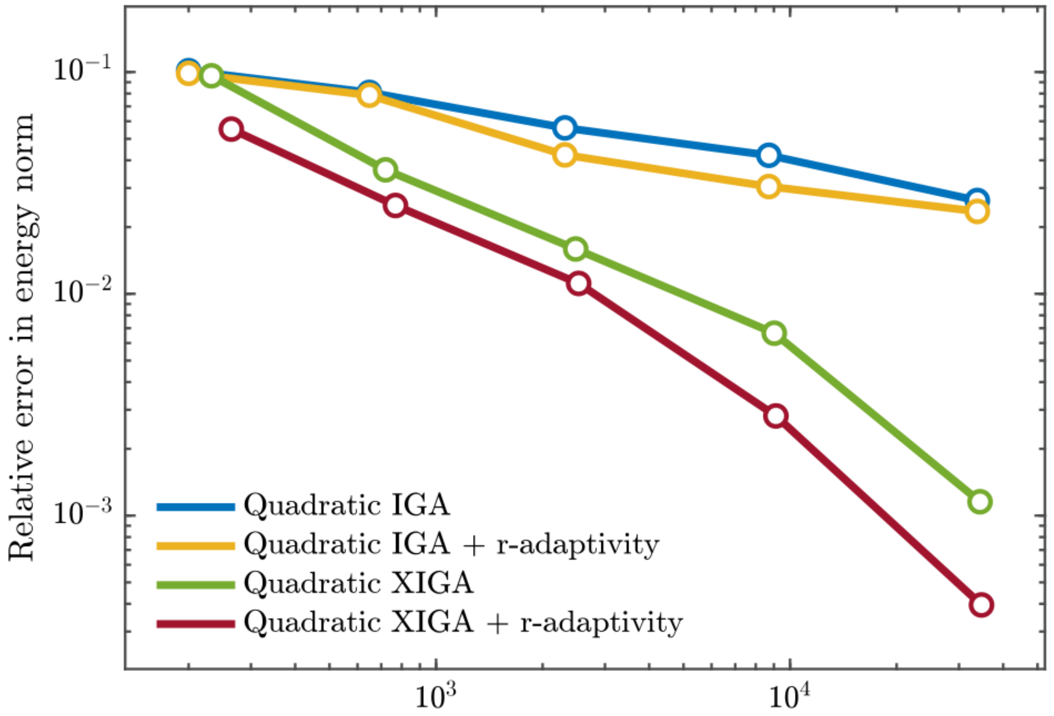
Inclusion r_1 = 0.4 in a matrix r_2 = 2.0; prescribed radial displacement on Γ_2.



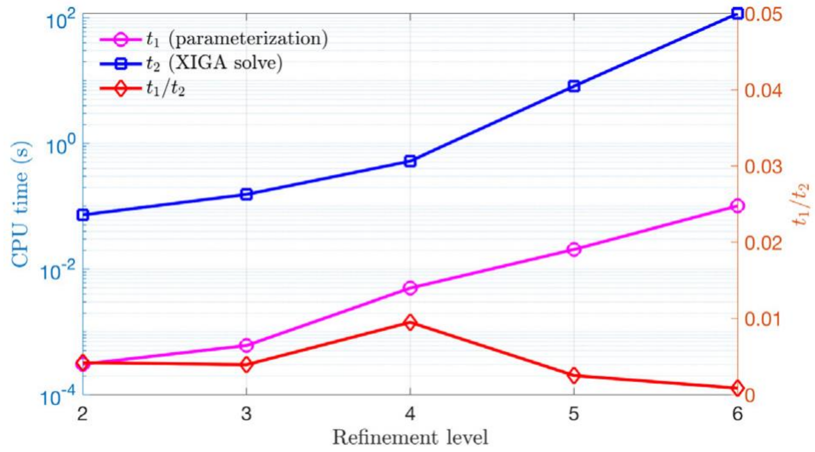
(a) analytical u_r · (b) r-adapted parameterization · (c) computed u_r · (d) absolute error — confined to a thin band at Γ.

- **Plane strain**; (E_1,ν_1)=(1, 0.25), (E_2,ν_2)=(10, 0.3).
- **Axisymmetric exact solution** from displacement + traction continuity.
- r-adaptation **concentrates iso-parameter lines on the interface**.
- **Computed field matches the reference; error is a thin interfacial band.**

Convergence and computational cost



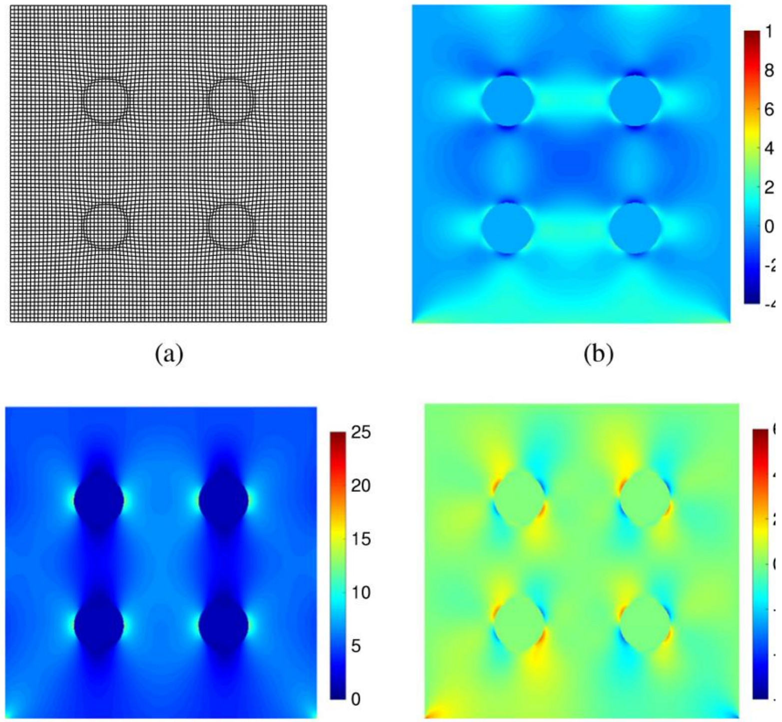
Energy-norm error vs. DOF. At fixed DOF, r-adaptive XIGA (red) is the most accurate; enrichment alone leaves a geometric-resolution error on the table.



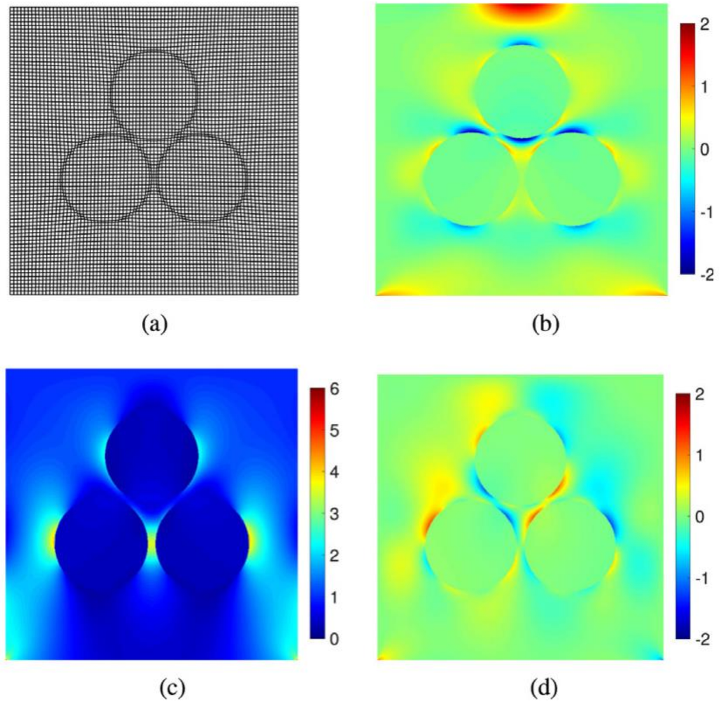
Parameterization time t_1 vs. XIGA solve t_2 (log scale) and their ratio t_1/t_2.

- At fixed DOF the energy-norm error is **reduced by up to 65.7%** relative to enrichment-only XIGA.
- The relocation adds **under 1%** to the overall cost; the convergence rate is preserved.

Multiple and closely-spaced interfaces



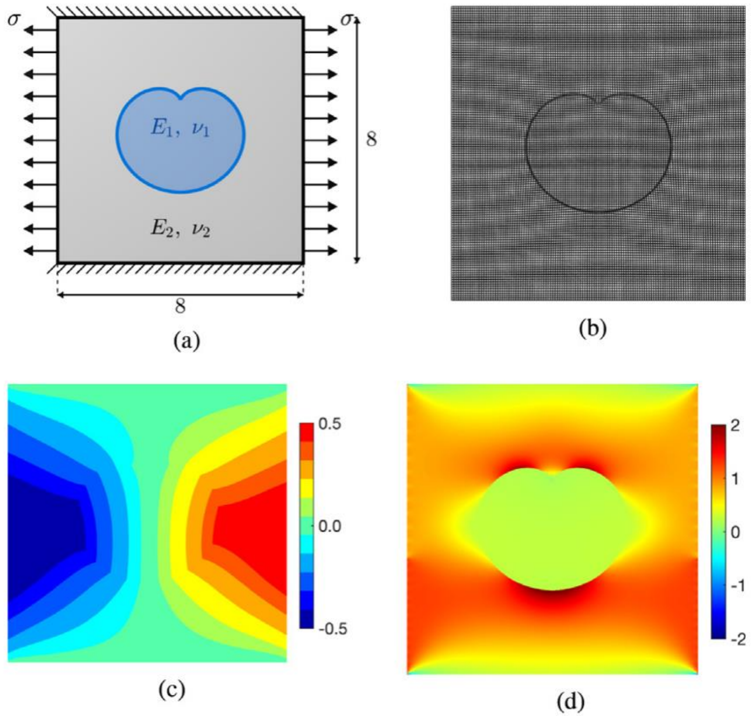
Four inclusions: (a) adapted net · (b–d) σ_{xx}, σ_{yy}, σ_{xy} — localized along boundaries, no oscillations.



Three close inclusions: control points pack into the narrow gaps where strain fields overlap.

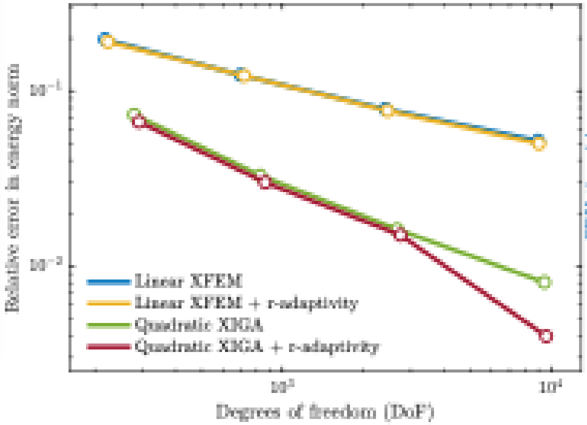
- Several interfaces impose **competing resolution demands** in a confined region.
- The aggregated monitor distributes points among neighbours — **no conflicting motions**.
- Benefit grows as interfaces get closer: **geometric resolution is the dominant error**.
- **Stable, oscillation-free stresses at fixed DOF — for both linear XFEM and quadratic XIGA.**

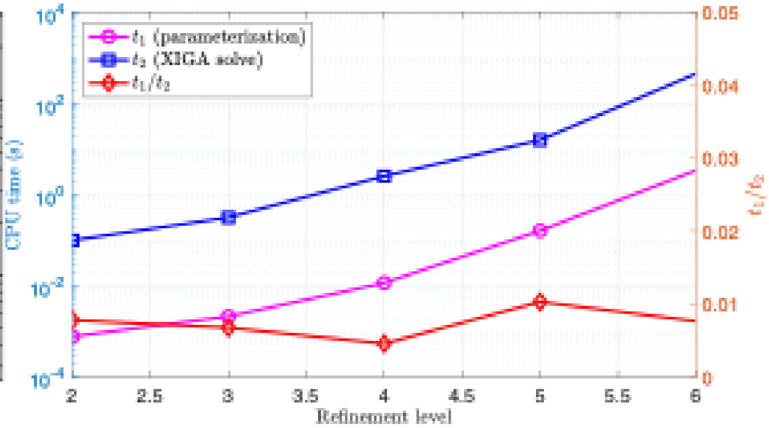
Cardioid inclusion with a geometric cusp



Heart-shaped inclusion under tension: (a) setup · (b) r-adapted net clustering at the cusp · (c) u_x · (d) σ_{xx} — sharp gradient and cusp concentration, no spurious oscillation.

- Non-convex interface with a **genuine geometric singularity** (tangent discontinuity at $\theta = \pi/2$).
- Redistribution is markedly non-uniform — denser along curvature and at the cusp — **with no manual intervention**.
- Such singular features can’t be resolved by **uniform refinement alone**.
- **Quadratic r-adaptive XIGA gains the most: relocating points toward the cusp restores the expected convergence rate.**





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Summary of findings

- Partition-of-unity enrichment and r-adaptivity are combined in a **single isogeometric framework** for weak-discontinuous problems.
- A KS-aggregated level-set monitor redistributes control points according to interface geometry, **rather than adding refinement**.
- CAD geometry, spline topology and global continuity are preserved, and the **degree-of-freedom count is left unchanged**.
- Each relocation step is a single symmetric positive-definite elliptic solve — **no nonlinear iteration**, with constant system size.
- Error is consistently reduced and the **expected convergence rate is recovered** across linear XFEM and quadratic XIGA.
- The benefit is most pronounced for **multiple, closely-spaced and singular interfaces**.

Key results

Energy-norm error
up to 65.7% lower at fixed DOF

Relocation cost
below 1% of the XIGA solve

Degrees of freedom
unchanged throughout

Convergence rate
recovered for all benchmarks

Future work

Into 3-D & real CAD

Extend to **three-dimensional problems and CAD-derived geometries** for realistic configurations.

Strong discontinuities

Combine r-adaptivity with continuity-reduction to treat **cracks and true field discontinuities**.

Fully automated adaptivity

Rigorous a-posteriori estimators and **automatic monitor-parameter selection**.

Thank you — questions welcome

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