Objective Function Terms

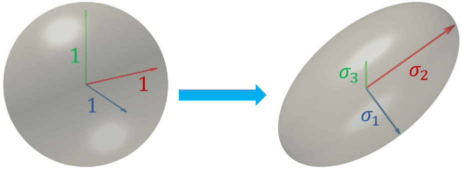
Angle Distortion

- Most Isometric ParametrizationS (MIPS) energy [HG2000, Fu+2015]:

$$\mathscr{E}^{\text{angle}} = \begin{cases} \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}, \\ \frac{1}{8}\left(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}\right)\left(\frac{\sigma_2}{\sigma_3} + \frac{\sigma_3}{\sigma_2}\right)\left(\frac{\sigma_1}{\sigma_3} + \frac{\sigma_3}{\sigma_1}\right), \end{cases}$$

where σ_i are the singular values of $\mathcal{J}$.

- Ideally, $\sigma_1 = \sigma_2 = \cdots = \sigma_d$.

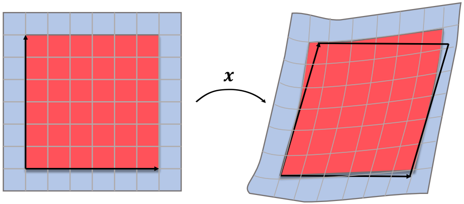


Area/Volume Distortion

- Area/volume distortion energy:

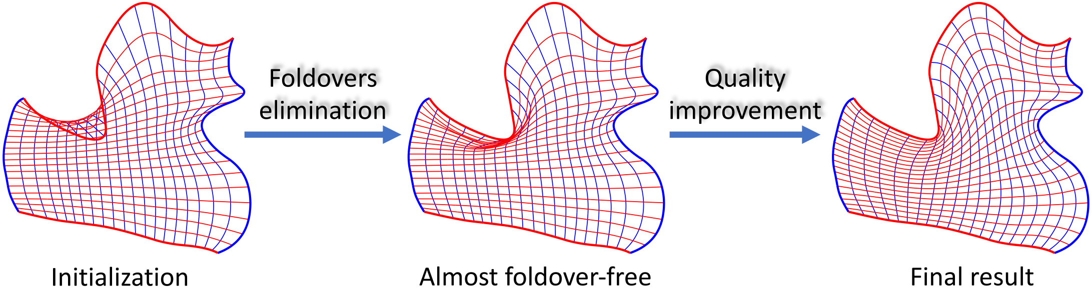
$$\mathscr{E}^{\text{unif.}}(\boldsymbol{x}) = \frac{|\mathcal{J}|}{vol(\Omega)} + \frac{vol(\Omega)}{|\mathcal{J}|},$$

where $vol(\Omega)$ denotes the area/volume of the computational domain Ω;

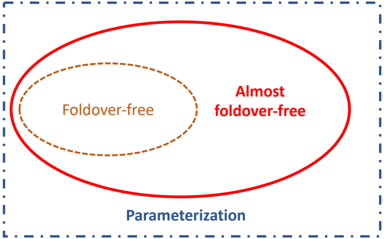


Barrier Function-based Parameterization Construction

- **Step I: Initialization**
  - Initialize using an algebraic method, such as the Coons Patch.
- **Step II: Foldovers Elimination**
  - Employ techniques to remove foldovers in the parameterization.
- **Step III: Quality Improvement**
  - Refine the parameterization to enhance its quality.



Foldovers Elimination: Almost Foldover-Free



- Solve the following optimization problem:

$$\mathop{\arg\min}_{\mathbf{P}_i,\ i\in\mathcal{I}_I} \quad E(\boldsymbol{x}) = \int_{\hat{\Omega}} \max\left(0, \delta - |\mathcal{J}|\right) \ \mathrm{d}\hat{\Omega},$$

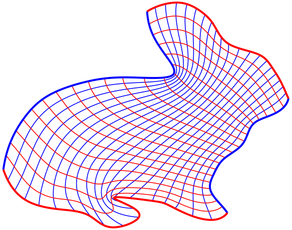
  where δ is a threshold ($\delta = 5\% vol(\Omega)$ as default).

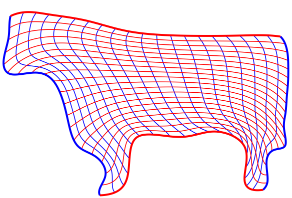
- Quality improvement (robustness consideration):

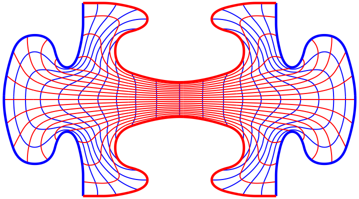
$$\mathscr{E}^{c} = \begin{cases} \int_{\hat{\Omega}} \left(\lambda_{\text{angle}}\mathscr{E}^{\text{angle}}(\boldsymbol{x}) + \lambda_{\text{vol}}\mathscr{E}^{\text{vol}}(\boldsymbol{x})\right) \ \mathrm{d}\hat{\Omega}, & \min|\mathcal{J}| > 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

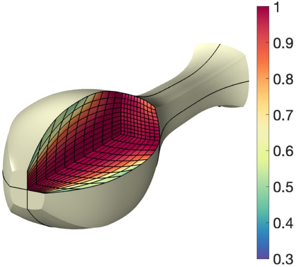
- Analytical gradient
  - for numerical stability and computational efficiency.

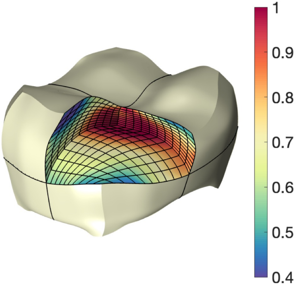
Gallery: Barrier Function-based Method

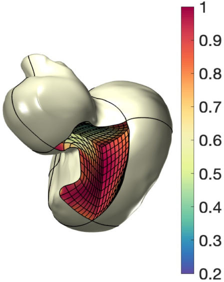






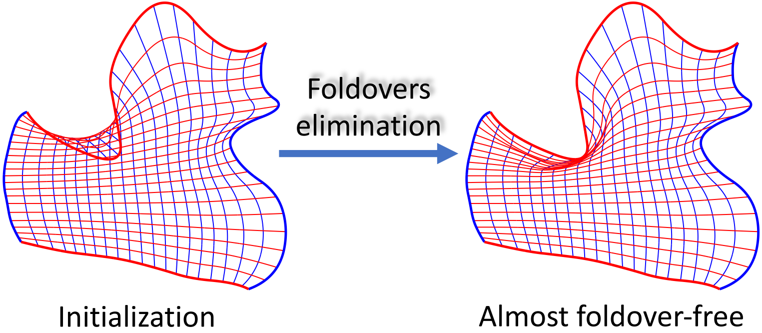






Penalty Function-based Parameterization Construction

- **Foldover elimination is often necessary but does not inherently improve parameterization quality.**
- Is it feasible to skip the foldover elimination step?



Penalty Function-based Parameterization Construction

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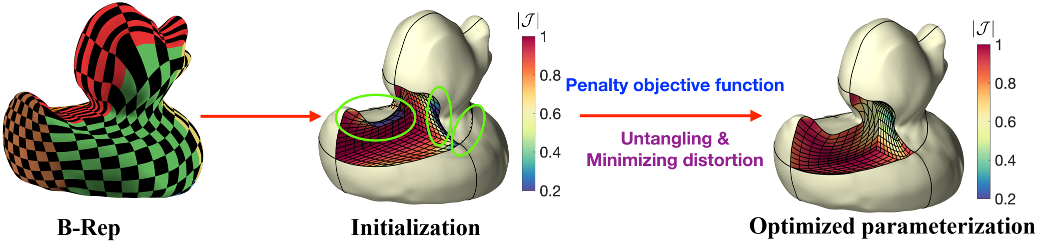
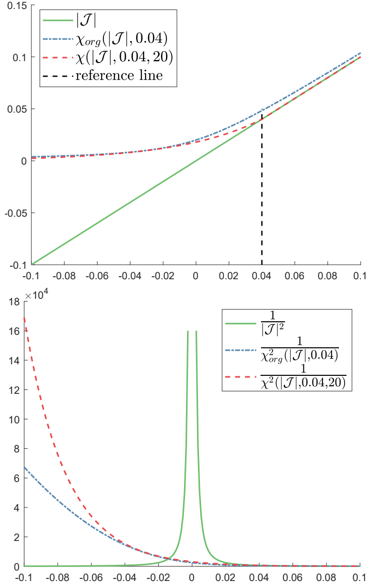


- **Certainly! Simultaneously untangling and minimizing distortion!!!**

Basic Idea: Penalty Function



- **Penalty function:**

$$\chi(|\mathcal{J}|, \varepsilon, \beta) = \begin{cases} \varepsilon \cdot e^{\beta(|\mathcal{J}|-\varepsilon)} & \text{if } |\mathcal{J}| \le \varepsilon \\ |\mathcal{J}| & \text{if } |\mathcal{J}| > \varepsilon \end{cases},$$

  where $\varepsilon > 0$ is small, and β is a penalty factor;

- $\chi(|\mathcal{J}|, \varepsilon, \beta)$ approaches ε for small $|\mathcal{J}|$, and equals $|\mathcal{J}|$ otherwise.

- $\frac{1}{\chi^2(|\mathcal{J}|,\varepsilon,\beta)}$:
  - **Penalizes negative Jacobians** with large values
  - **Accepts positive Jacobians** with small values

Jacobian Regularization & Revised Objective Function

- **Only one optimization problem needs to be solved:**

$$\begin{aligned}\mathop{\arg\min}_{\mathbf{P}_i,\ i\in\mathcal{I}_I} \mathscr{E}^c &= \int_{\hat{\Omega}} \left(\lambda_{\text{mips}}\mathscr{E}^c_{\text{mips}} + \lambda_{\text{vol}}\mathscr{E}^c_{\text{vol}}\right)\, \mathrm{d}\hat{\Omega} \\ &= \int_{\hat{\Omega}} \left(\frac{\lambda_{\text{mips}}}{8}\kappa_F^2(\boldsymbol{\mathcal{J}}) \cdot \frac{|\boldsymbol{\mathcal{J}}|^2}{\chi^2(|\boldsymbol{\mathcal{J}}|,\varepsilon,\beta)} + \lambda_{\text{vol}} \left(\frac{vol(\Omega)}{\chi(|\boldsymbol{\mathcal{J}}|,\varepsilon,\beta)} + \frac{\chi(|\boldsymbol{\mathcal{J}}|,\varepsilon,\beta)}{vol(\Omega)} \right)\right)\, \mathrm{d}\hat{\Omega},\end{aligned}$$

where $\mathbf{P}_i,\ i \in \mathcal{I}_I$ are the unknown inner control points.

- **Computational Techniques:**
  - Analytical gradient for numerical stability and efficiency;
  - Reduced numerical integration complexity;
  - Pre-computation for faster calculations.