Analysis-suitable Parameterization Construction and Curvature-based r-Adaptive Parameterization for IsoGeometric Analysis

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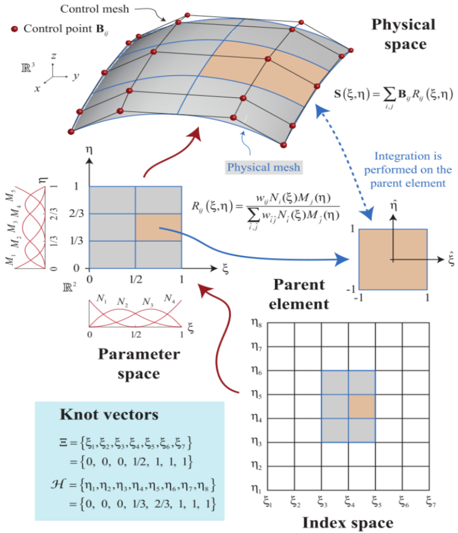
Contributors

- Ye Ji et al., Constructing high-quality ..., Journal of Computational and Applied Mathematics, 396 (2021), 113615.
- Ye Ji et al., Penalty function-based volumetric ..., Computer Aided Geometric Design, 94 (2022), 102075.
- Ye Ji et al., Curvature-Based r-Adaptive ..., Computer-Aided Design, 150 (2022), 103305.

Catalogue

IsoGeometric Analysis (IGA)



Source: Figure from [Cottrell et al. 2009]

- Proposed by T.J.R. Hughes et al., 2005.
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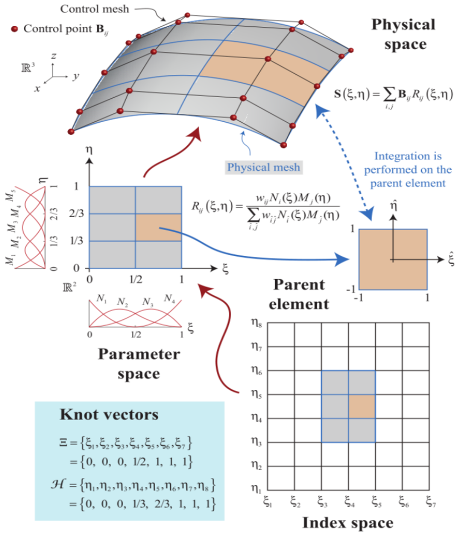

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- Advantages:
  - Integration of design and analysis;
  - Exact and efficient geometry;
  - No data type transition and mesh generation;
  - Simplified mesh refinement;
  - High order continuous field;
  - Superior approximation properties.

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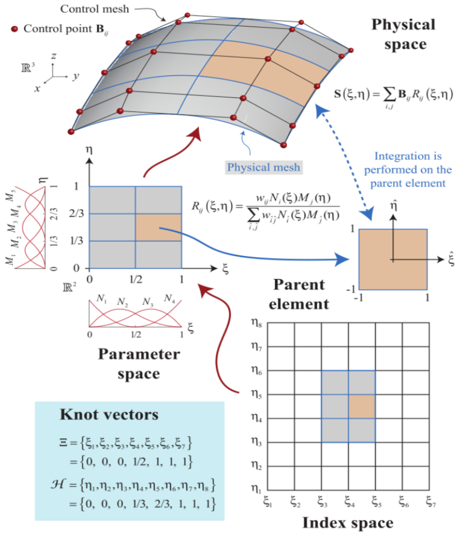

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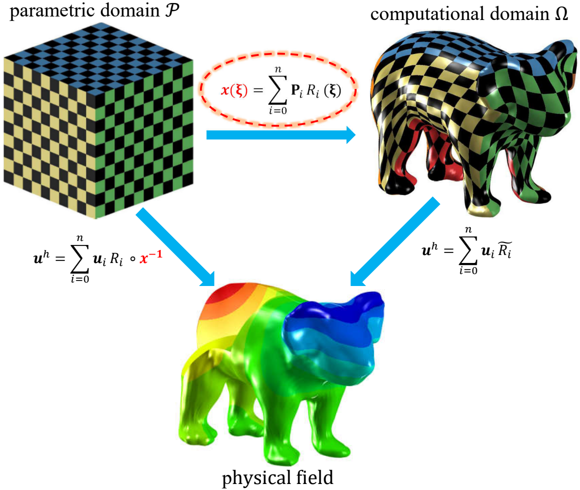
  - No data type transition and mesh generation;
  - Simplified mesh refinement;
  - High order continuous field;
  - Superior approximation properties.
- Very broad applications: such as shell analysis, fluid-structure interaction, and shape and topology optimization.

Research motivation



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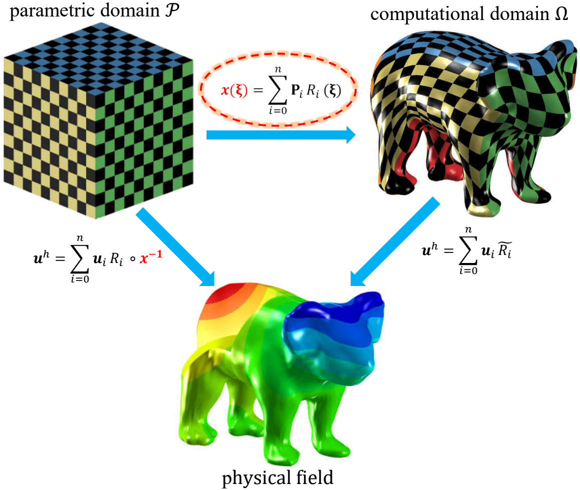

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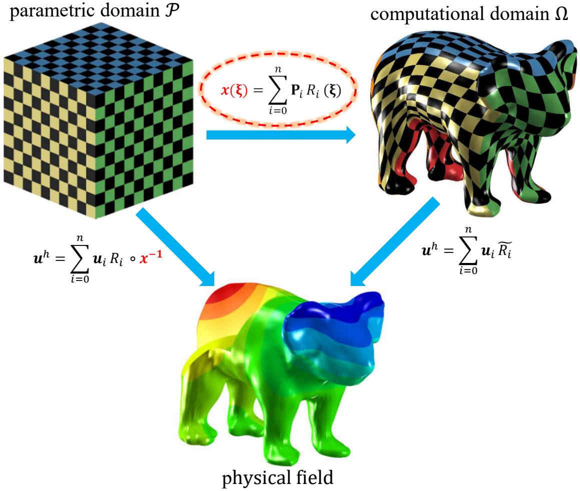


- Most modern CAD systems only focus on boundary representations (B-Reps) in geometry modeling.
- **Problem statement**:
  - From a given B-Rep, constructing an analysis-suitable parameterization $\boldsymbol{x}$.
  - Analysis-suitable parameterizations should
    - be **bijective**;
    - ensure as **low angle and volume distortion** as possible.

Catalogue

Related work - planar parameterization

- Crucial influence of parameterization quality on subsequent analysis: Cohen+2010, Xu+2013a, Pilgerstorfer+2014.
- Planar domain parameterization:
  - Single-patch:
    - Algebraic methods: discrete Coons method [Farin and Hansford 1999], linear methods [Gravesen+2012];
    - Constrained optimization methods: Xu+2011, Gravesen+2014, Ugalde+2018;
    - Variation harmonic mapping [Xu+2013b], PDE-based method [Hinz+2018], Teichmüller mapping [Nian and Chen 2016], low-rank quasi-conformal method [Pan+2018], large elastic deformation method [Shamanskiy+2020];
    - Barrier function method [Ji+2021];
    - Jacobian regularization technique [Garanzha+1999 2021, Wang and Ma 2021].
  - Multi-patch: Xu+2015, Buchegger+2018, Xu+2018, Xiao+2018, Kapl+2017a 2017b 2018 2019, Blidia+2020, Bastl and Slabá 2021, Wang+2022.

Related work - volumetric parameterization

- Compared with the planar problem, constructing analysis-suitable volumetric parameterizations is more challenging both geometrically and computationally.
- Single-block:
  - Constrained optimization methods: Xu+2013c 2017, Wang and Qian 2014
    Suffer from computing huge amounts of constraints (impractical for large-scale problems);
  - Spline fitting methods: Martin+2009, Lin+2015, Liu+2020, Yuan+2021
    Need mesh generation of the discretized computational domains;
  - Barrier function methods: Pan and Chen 2019, Pan+2020
    Need an already bijective initialization which is usually difficult to obtain.
- Multi-block: Xu+2013 2017, Lin+2018, Chen+2019 2022, Haberleitner+2019.
- Non-standard B-splines or NURBS: such as C^1 Powell-Sabin splines, toric patches, THB-splines, T-splines, PHT-splines, and Catmull-Clark volumetric subdivision.

Catalogue

Problem statement

- A spline-based parameterization $\boldsymbol{x}$ from a parametric domain $\mathcal{P} = [0,1]^d$ $(d = 2, 3)$ to computational domain Ω is of the following form

$$\boldsymbol{x}(\boldsymbol{\xi}) = \mathbf{R}^{\mathrm{T}}\mathbf{P} = \underbrace{\sum_{i \in \mathcal{I}_I} \mathbf{P}_i R_i(\boldsymbol{\xi})}_{\text{unknown}} + \underbrace{\sum_{j \in \mathcal{I}_B} \mathbf{P}_j R_j(\boldsymbol{\xi})}_{\text{known}}, \tag{1}$$

  where $\mathbf{P}_i$ are unknown inner control points and $\mathbf{P}_j$ are given boundary control points.
- **GOAL:** To construct the **unknown inner control points $\mathbf{P}_i$** such that $\boldsymbol{x}$ is **bijective** and has the **lowest possible angle and area/volume distortion**.

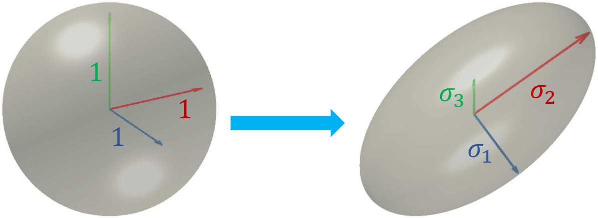
Objective function: angle distortion

- Most-Isometric ParameterizationS (MIPS) energy [Hormann and Greiner 2000, Fu+2015]:

$$E_{\text{angle}}(\boldsymbol{x}) = \begin{cases} \dfrac{\sigma_1}{\sigma_2} + \dfrac{\sigma_2}{\sigma_1}, & 2D, \\ \dfrac{1}{8}\left(\dfrac{\sigma_1}{\sigma_2} + \dfrac{\sigma_2}{\sigma_1}\right)\left(\dfrac{\sigma_2}{\sigma_3} + \dfrac{\sigma_3}{\sigma_2}\right)\left(\dfrac{\sigma_1}{\sigma_3} + \dfrac{\sigma_3}{\sigma_1}\right), & 3D. \end{cases} \tag{2}$$

  where σ_i are the singular values of the Jacobian matrix $\mathcal{J}$ of the parameterization $\boldsymbol{x}$.

- When $\sigma_1 = \sigma_2 = \ldots = \sigma_d$, $\boldsymbol{x}$ is conformal and E_{angle} reaches its minimum value.

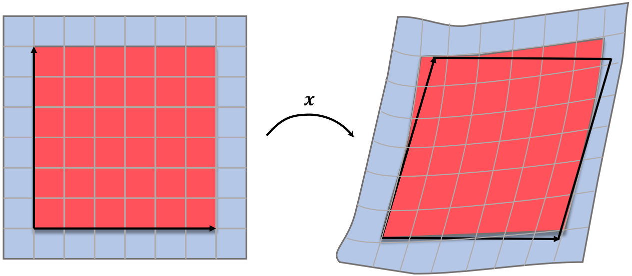


Objective function: area/volume distortion

- Area/volume distortion energy:

$$E_{\text{vol}}(\boldsymbol{x}) = \frac{|\mathcal{J}|}{vol(\Omega)} + \frac{vol(\Omega)}{|\mathcal{J}|}, \tag{3}$$

where $vol(\Omega)$ denotes the area/volume of the computational domain Ω;



Objective function: variational formulation

- **Basic idea:** to solve the following constrained optimization problem:

$$\begin{aligned} \underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min} \quad & E(\boldsymbol{x}) = \int_{\mathcal{P}} \left(\lambda_1 E_{\text{angle}}(\boldsymbol{x}) + \lambda_2 E_{\text{vol}}(\boldsymbol{x})\right) \, \mathrm{d}\mathcal{P}, \\ s.t. \quad & \boldsymbol{x} \text{ is bijective.} \end{aligned} \tag{4}$$

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- Suppose that the given B-Rep is bijective. $\boldsymbol{x}$ is bijective $\Leftrightarrow |\mathcal{J}(\boldsymbol{x}(\boldsymbol{\xi}))| \neq 0,\ \forall \boldsymbol{\xi} \in \mathcal{P}$.

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- Suppose that the given B-Rep is bijective. $\boldsymbol{x}$ is bijective $\Leftrightarrow |\mathcal{J}(\boldsymbol{x}(\boldsymbol{\xi}))| \neq 0,\ \forall \boldsymbol{\xi} \in \mathcal{P}$.
- Due to the high-order continuity of $\boldsymbol{x}$, we need $|\mathcal{J}| > 0\ (< 0),\ \forall \boldsymbol{\xi} \in \mathcal{P}$.

Treatment of bijectivity constraint

- The Jacobian determinant can be represented by a linear combination of splines

$$|\mathcal{J}| = \sum_i |\mathcal{J}|_i N_i(\boldsymbol{\xi}) \tag{5}$$

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- However, the number of the constraints can be huge. [Pan et al. 2020, Ji et al. 2021].
(To a bi-cubic planar NURBS parameterization with 20×20 control points, the number of inequality constraints is over 34**k**.)

Equivalence problem: unconstrained optimization

- Recall the planar MIPS energy,

$$E_{angle}^{2D}(\boldsymbol{x}) = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2} = \frac{trace(\mathcal{J}^{\mathrm{T}}\mathcal{J})}{|\mathcal{J}|}.$$

Since the Jacobian determinant appears in its denominator, it proceeds to infinity if the Jacobian determinant $|\mathcal{J}|$ approaches zero.

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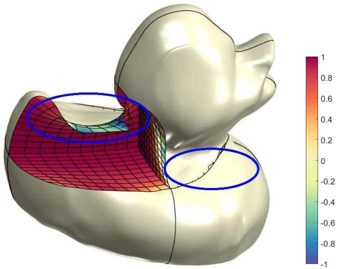
- Remove the constraints and solve the following **unconstrained optimization problem**:

$$\arg\min_{\mathbf{P}_i,\ i\in\mathcal{I}_I} \quad E(\boldsymbol{x}) = \int_{\mathcal{P}} \left(\lambda_1 E_{\text{angle}}(\boldsymbol{x}) + \lambda_2 E_{\text{vol}}(\boldsymbol{x})\right) \, \mathrm{d}\mathcal{P}. \tag{6}$$

Initialization

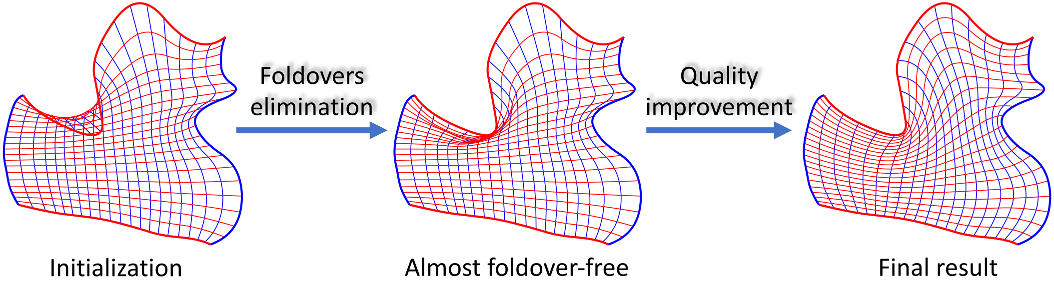


Initial parameterization.

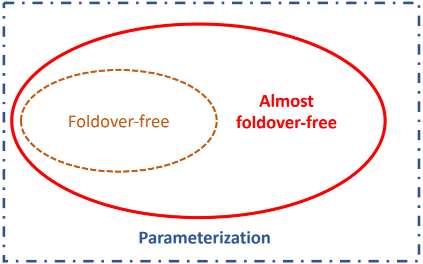
- Many algebraic methods can be adopted to initialize:
  - Discrete Coon's patch [Farin and Hansford 1999];
  - Spring patch [Gravesen et al. 2012];
  - Smoothness energy minimization [Wang et al. 2003, Pan et al. 2020];
  - ...
- No guarantee of bijectivity.
- However, an already bijective parameterization is needed in our optimization problem (6).

Barrier function-based parameterization construction

- Three-step strategy.



Foldovers elimination: almost foldover-free parameterization



- Some works solve the following Max-Min problem:

$$\arg\min_{\mathbf{P}_i,\ i\in\mathcal{I}_I} \max_j |\mathcal{J}|_j,$$

where $|\mathcal{J}|_j$ are the expansion coefficients of $|\mathcal{J}|$.

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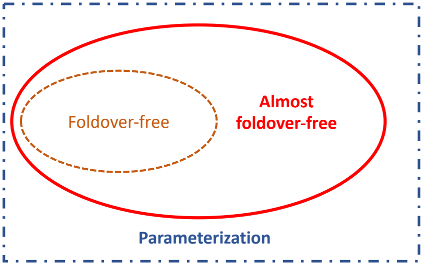

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- **High computational costs still but NOT necessary!**

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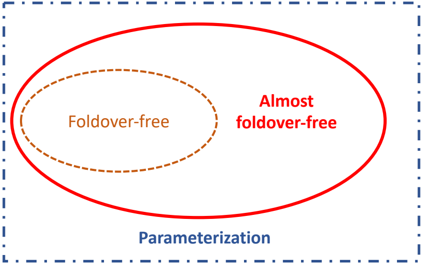

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- **High computational costs still but NOT necessary!**
- We solve the following problem instead:

$$\underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min}\quad E(\boldsymbol{x}) = \int_{\mathcal{P}} \max\left(0, \delta - |\mathcal{J}|\right)\ \mathrm{d}\mathcal{P},$$

where δ is a threshold ($\delta = 5\% vol(\Omega)$ as default).

Quality improvement: robustness consideration

- Recall that E_{angle} proceeds to infinity if the Jacobian determinant $\mathcal{J}$ approaches zero.
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- Line search ensures sufficient reduction, e.g., strong Wolfe condition.

Quality improvement: robustness consideration

- Recall that E_{angle} proceeds to infinity if the Jacobian determinant $\mathcal{J}$ approaches zero.
- DANGER!: discontinuous function value change in numerical optimization.
- Line search ensures sufficient reduction, e.g., strong Wolfe condition.
- With this feature, we simply revise the objective function (barrier function):

$$E^c = \begin{cases} \int_{\mathcal{P}} \left(\lambda_1 E_{\text{angle}}(\boldsymbol{x}) + \lambda_2 E_{\text{vol}}(\boldsymbol{x})\right) \, \mathrm{d}\mathcal{P}, & \text{if } \min|\mathcal{J}| > 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Analytical gradient: for stability aspect

- Many optimization solvers have the option to approximate the gradient by numerical differentiation, e.g., the following high-order scheme

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x+2h)}{12h} + \frac{h^4}{30} f^{(5)}(c),$$

where $c \in [x-2h, x+2h]$.

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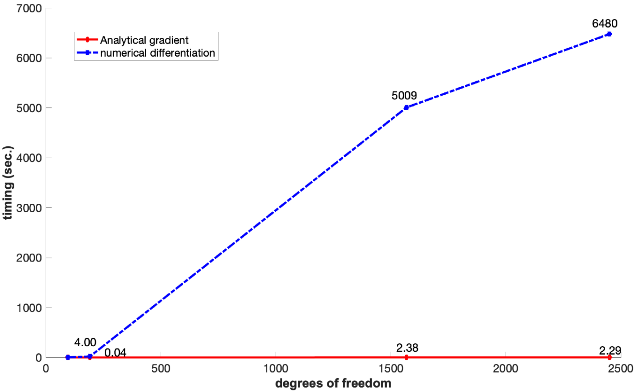
where $c \in [x-2h, x+2h]$.

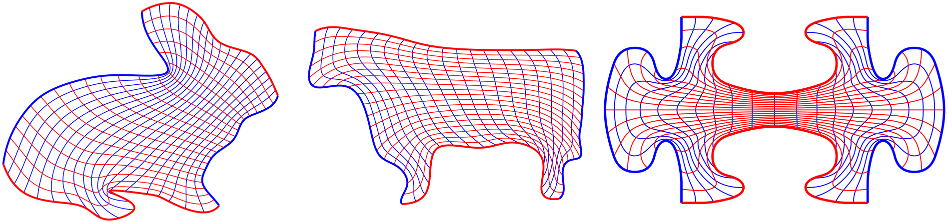
- Hard to select a suitable step size h, especically for our problem.

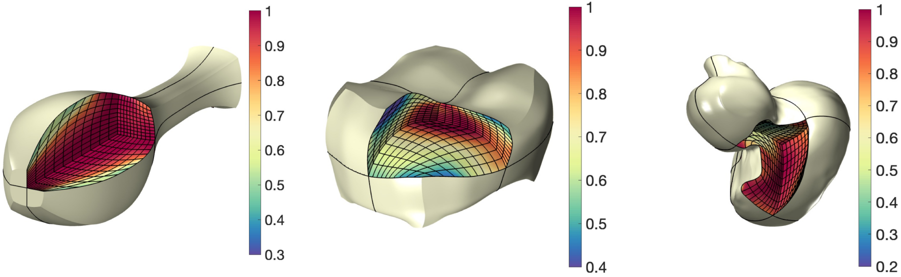
Analytical gradient: for efficiency aspect

- To a single-patch tri-cubic B-spline parameterization with 25 control points along each direction (using standard Gauss quadrature rule), $4 * 23^3 * (3+1)^3 > 3$ M function evaluations are performed for once line-search.



Gallery: barrier function-based method





Problem with the basic objective function

- Recall the MIPS energy

$$E_{\text{mips}} = \frac{1}{8}\left(\frac{\sigma_1}{\sigma_2}+\frac{\sigma_2}{\sigma_1}\right)\left(\frac{\sigma_2}{\sigma_3}+\frac{\sigma_3}{\sigma_2}\right)\left(\frac{\sigma_1}{\sigma_3}+\frac{\sigma_3}{\sigma_1}\right) = \frac{1}{8}\left(\frac{\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)\left(\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2\right)}{|\mathcal{J}|^2}-1\right); \tag{7}$$

- The Jacobian determinant $|\mathcal{J}|$ appears in the denominator, which forms a barrier and suppresses foldovers;
- However, the **prerequisite is to find an already bijective initialization**, which is difficult to obtain efficiently for complex computational domains;

Problem with the basic objective function

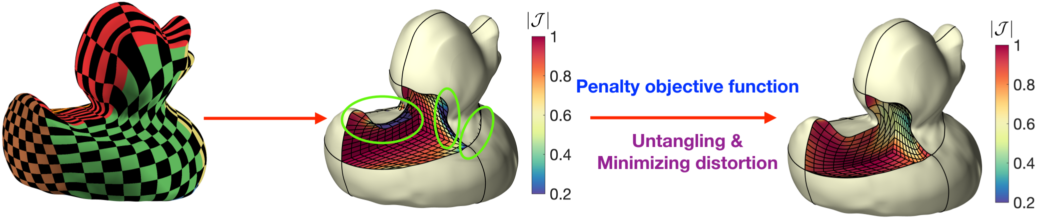
- Recall the MIPS energy

$$E_{\text{mips}} = \frac{1}{8}\left(\frac{\sigma_1}{\sigma_2}+\frac{\sigma_2}{\sigma_1}\right)\left(\frac{\sigma_2}{\sigma_3}+\frac{\sigma_3}{\sigma_2}\right)\left(\frac{\sigma_1}{\sigma_3}+\frac{\sigma_3}{\sigma_1}\right) = \frac{1}{8}\left(\frac{\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)\left(\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2\right)}{|\mathcal{J}|^2}-1\right); \quad (7)$$

- The Jacobian determinant $|\mathcal{J}|$ appears in the denominator, which forms a barrier and suppresses foldovers;
- However, the **prerequisite is to find an already bijective initialization**, which is difficult to obtain efficiently for complex computational domains;
- **The foldovers elimination does not improve sufficient to the parameterization quality.**

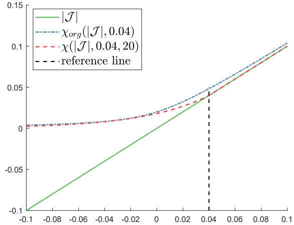
Penalty function-based parameterization construction

- Avoids extra foldovers elimination steps.
- **Untangling and minimizing distortion perform simultaneously!!!**

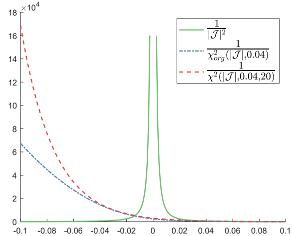


B-Rep **Initialization** **Optimized parameterization**

Basic idea: Penalty function



$\chi(|\boldsymbol{\mathcal{J}}|, \varepsilon, \beta)$



$\frac{1}{\chi^2(|\boldsymbol{\mathcal{J}}|,\varepsilon,\beta)}$

- **Penalty function:**

$$\chi(|\boldsymbol{\mathcal{J}}|, \varepsilon, \beta) = \begin{cases} \varepsilon \cdot e^{\beta(|\boldsymbol{\mathcal{J}}|-\varepsilon)} & \text{if } |\boldsymbol{\mathcal{J}}| \leq \varepsilon \\ |\boldsymbol{\mathcal{J}}| & \text{if } |\boldsymbol{\mathcal{J}}| > \varepsilon \end{cases}, \quad (8)$$

  where ε is a small positive number and β is a penalty factor;

- $\chi(|\boldsymbol{\mathcal{J}}|, \varepsilon, \beta)$ equals a small positive number if $|\boldsymbol{\mathcal{J}}| < \varepsilon$, and strictly equals the Jacobian determinant $|\boldsymbol{\mathcal{J}}|$ if $|\boldsymbol{\mathcal{J}}| \geq \varepsilon$;

- $\frac{1}{\chi^2(|\boldsymbol{\mathcal{J}}|,\varepsilon,\beta)}$ have **very large values to penalize the negative Jacobians and small values to accept positive Jacobians**.

Jacobian regularization and revised objective function

- With this basic idea, we solve the following optimization problem:

$$\begin{aligned}\underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min} E^c &= \int_{\mathcal{P}} \left(\lambda_1 E^c_{\text{mips}} + \lambda_2 E^c_{\text{vol}}\right)\, \mathrm{d}\mathcal{P} \\ &= \int_{\mathcal{P}} \left(\frac{\lambda_1}{8}\kappa_F^2(\mathcal{J}) \cdot \frac{|\mathcal{J}|^2}{\chi^2(|\mathcal{J}|,\varepsilon,\beta)} + \lambda_2\left(\frac{vol(\Omega)}{\chi(|\mathcal{J}|,\varepsilon,\beta)} + \frac{\chi(|\mathcal{J}|,\varepsilon,\beta)}{vol(\Omega)}\right)\right)\, \mathrm{d}\mathcal{P},\end{aligned} \tag{9}$$

where $\mathbf{P}_i$, $i \in \mathcal{I}_I$ are the unknown inner control points.

- Now, **only one optimization problem is solved.**

Analytical gradient computation

- During the gradient-based optimization process, an analytical gradient calculation is very important for efficiency and stability;
- Through the chain rule, we have

$$\partial_p \kappa_F^2(\mathcal{J}) = 2\ Tr((\|\mathcal{J}^{-1}\|_F^2 \mathcal{J}^{\mathrm{T}} - \|\mathcal{J}\|_F^2 (\mathcal{J}\mathcal{J}^{\mathrm{T}}\mathcal{J})^{-1})\partial_p \mathcal{J}). \tag{10}$$

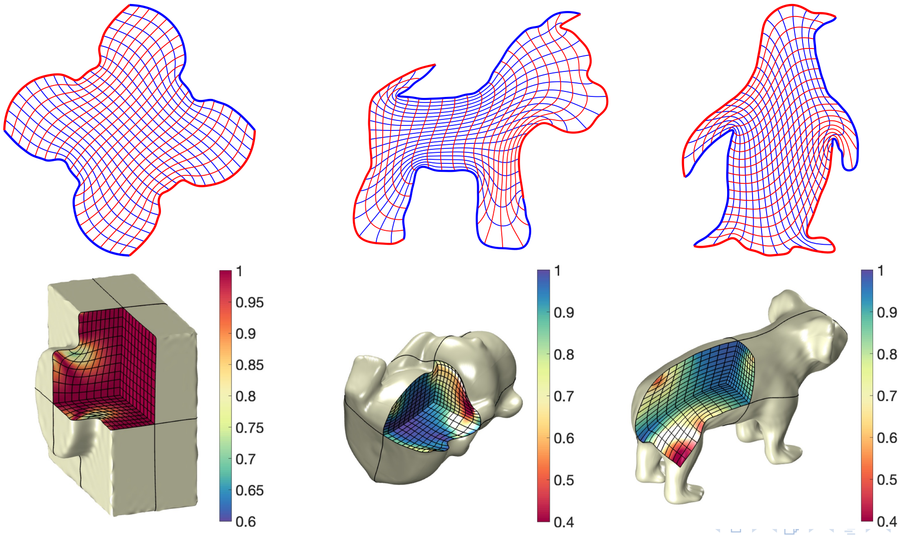
and

$$\partial_p \kappa_{F,\varepsilon}^2(\mathcal{J}) = \frac{\partial_p \kappa_F^2(\mathcal{J})|\mathcal{J}|^2 + 2\kappa_F^2(\mathcal{J})|\mathcal{J}|\partial_p|\mathcal{J}|}{\chi^2} - 2\kappa_{F,\varepsilon}^2(\mathcal{J})\frac{\partial \chi}{\partial|\mathcal{J}|}\frac{\partial_p|\mathcal{J}|}{\chi}; \tag{11}$$

- Eventually, we obtain the partial derivatives of the corrected objective function

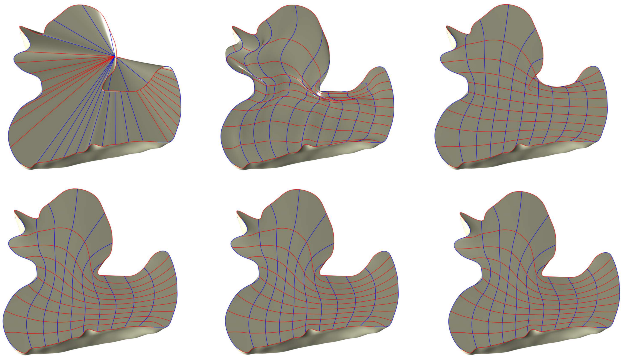
$$\partial_p \kappa_{F,\varepsilon}^2(\mathcal{J}) = \frac{\partial_p \kappa_F^2(\mathcal{J})|\mathcal{J}|^2 + 2\kappa_F^2(\mathcal{J})|\mathcal{J}|\partial_p|\mathcal{J}|}{\chi^2} - 2\kappa_{F,\varepsilon}^2(\mathcal{J})\frac{\partial \chi}{\partial|\mathcal{J}|}\frac{\partial_p|\mathcal{J}|}{\chi}. \tag{12}$$

Gallery: penalty function-based results



Catalogue

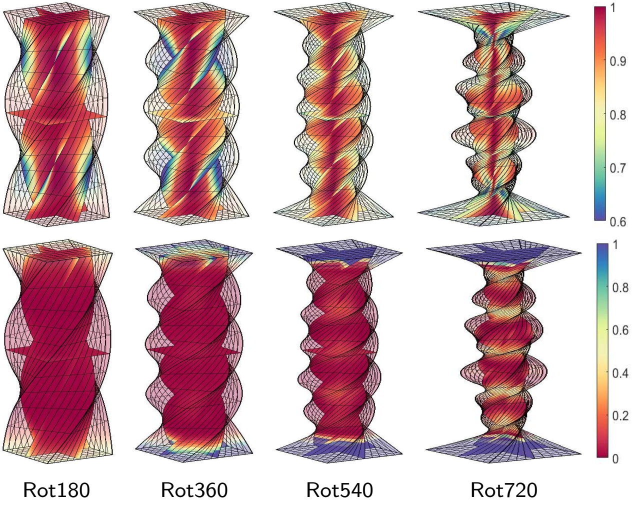
Parameterization results from different initialization methods



Same point Discrete Coons Smoothness energy

- The resulting parameterizations are almost the same from different initializations.
- It means our method converges to the same minimum and is insensitive to different initializations.

Robustness test



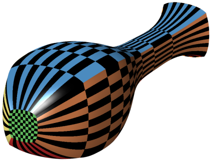
- Rotated cuboids parameterized by tri-cubic NURBS solids.
- Quality metrics:
  - **Scaled Jacobian** (optimal value 1):

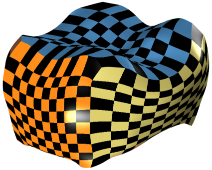
$$m_{SJ} = \frac{|\mathcal{J}|}{\|\mathbf{x}_{,\xi_1}\| \cdot \|\mathbf{x}_{,\xi_2}\| \cdot \|\mathbf{x}_{,\xi_3}\|}.$$

  - **Uniformity metric** (optimal value 0):

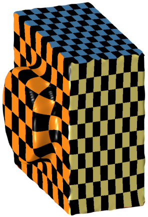
$$m_{unif.} = (\frac{|\mathcal{J}|}{vol(\Omega)} - 1)^2.$$

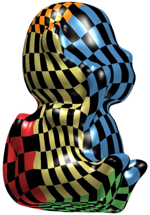
Six more complicated models


Vase


Tooth

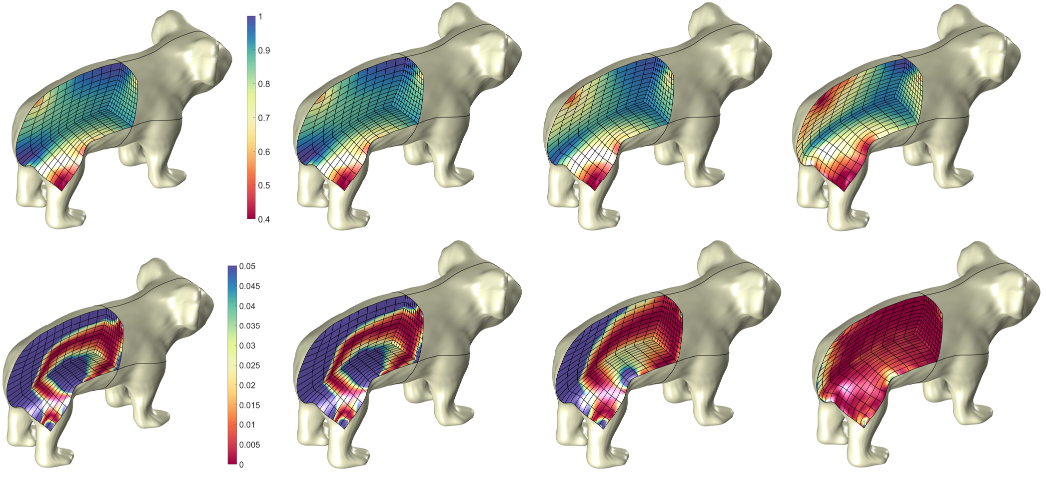

Duck


Component


Monkey

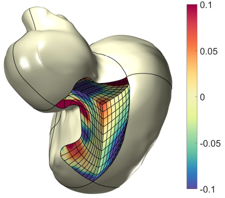

Koala

Influence of different proportions of parameters λ_1 and λ_2

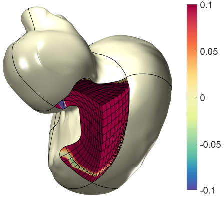


$\lambda_1 = 4,\ \lambda_2 = 0$ $\lambda_1 = 4,\ \lambda_2 = 1$ $\lambda_1 = 4,\ \lambda_2 = 10$ $\lambda_1 = 4,\ \lambda_2 = 100$

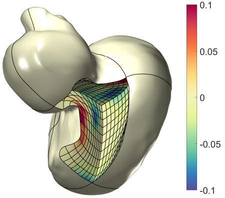
Comparison: Our method vs. current competitive approaches



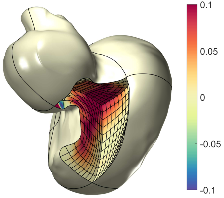
$m_{SJ}^{\text{Algo. 1}} - m_{SJ}^{\text{Pan}}$



$m_{unif.}^{\text{Pan}} - m_{unif.}^{\text{Algo. 1}}$



$m_{SJ}^{\text{Algo. 1}} - m_{SJ}^{\text{Liu}}$

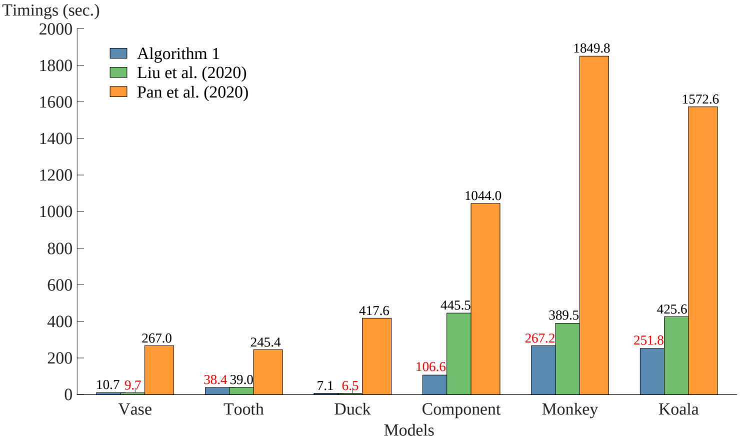


$m_{unif.}^{\text{Liu}} - m_{unif.}^{\text{Algo. 1}}$

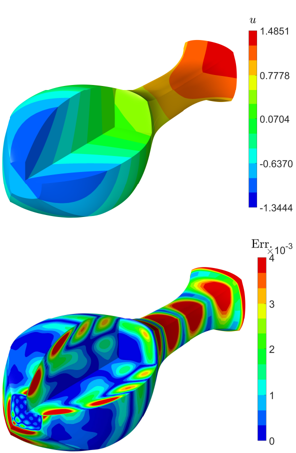
- We compare our method with two current competitors, i.e., Pan et al. 2020 and Liu et al. 2020.
- Positive values (red regions) indicate our method has lower angle distortion and/or lower volume distortion.

Efficiency: Our method vs. current competitive approaches

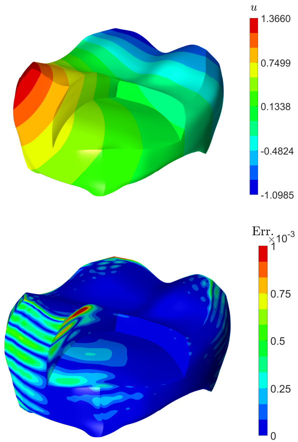
- Our method $\gg$ Pan et al. (2020);
- First three small-scale models, our method $\approx$ Liu et al. (2020);
- Last three large-scale models, our method $>$ Liu et al. (2020).



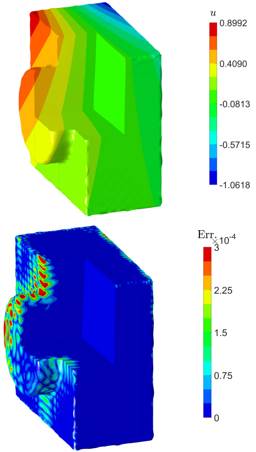
Application to IGA simulation: Poisson's problem



Vase, Err. $= 1.717 \times 10^{-3}$

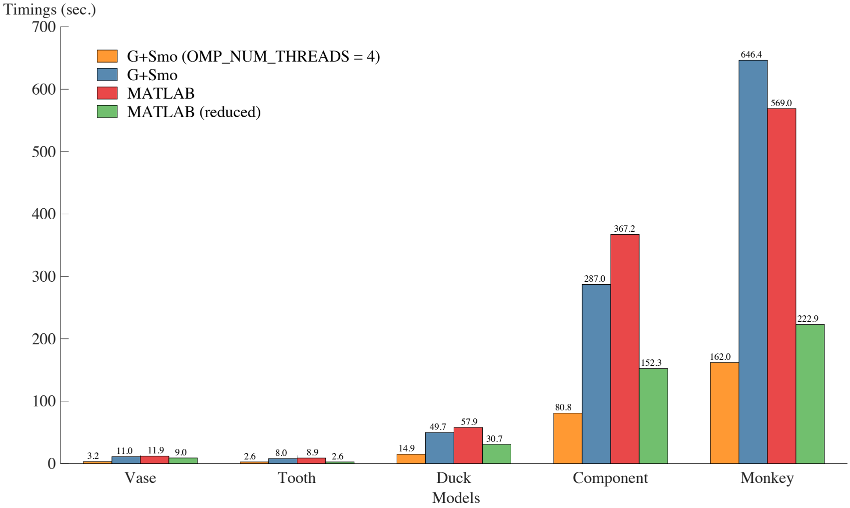


Tooth, Err. $= 1.944 \times 10^{-4}$

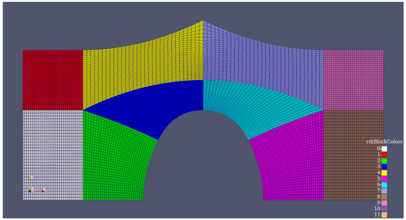


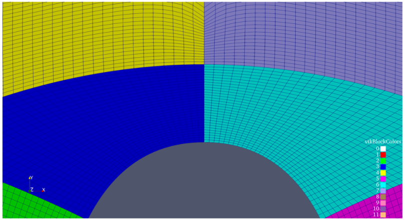
Component, Err. $= 2.824 \times 10^{-5}$

G+Smo implementation with OPENMP

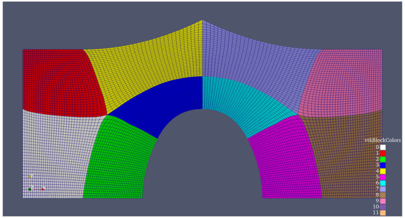


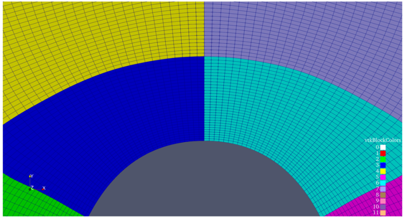
Multi-patch result: multipatch_tunnel.xml





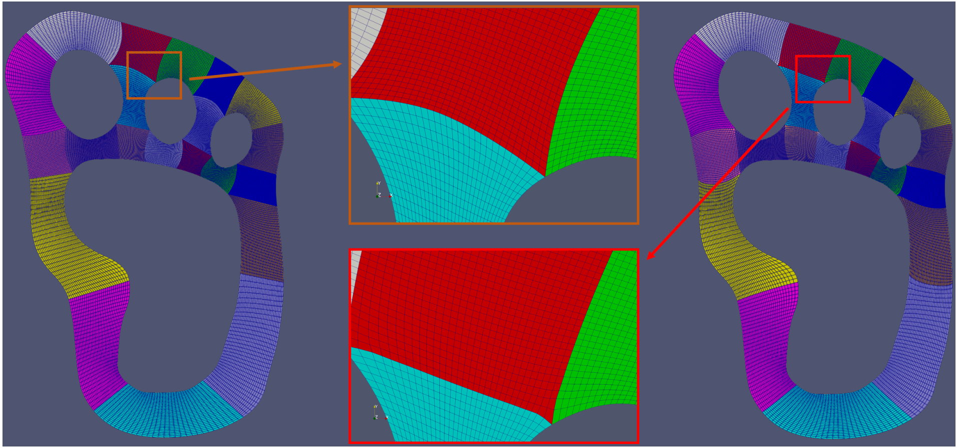
Fixed interfaces



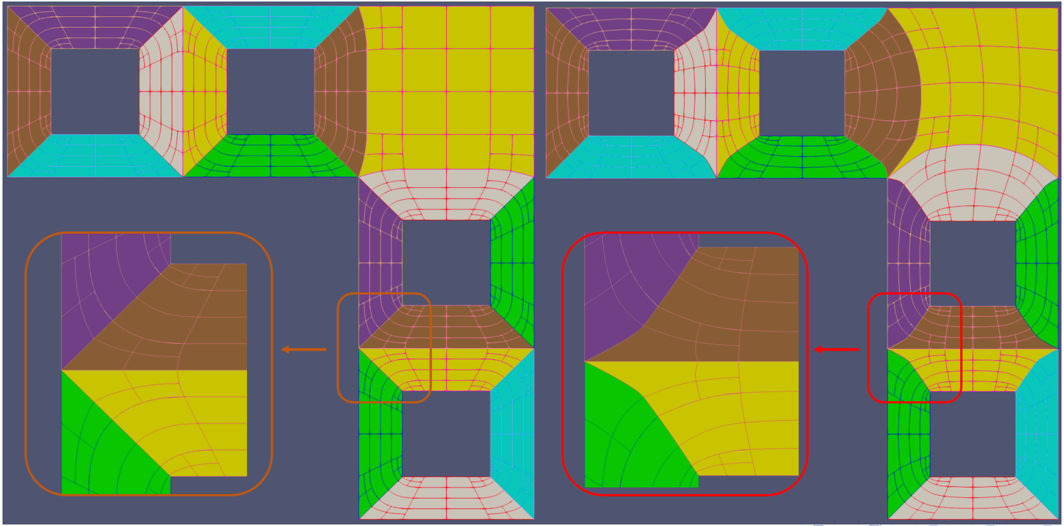


Free interfaces

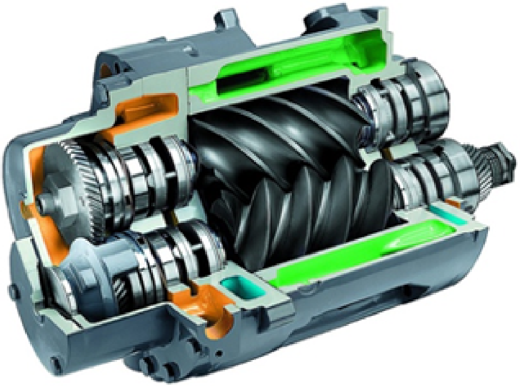
Multi-patch result: `yeti_footprint.xml`

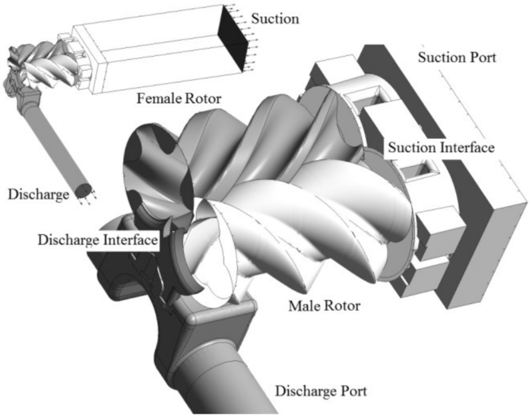


Compatible to multi-patch THB parameterization

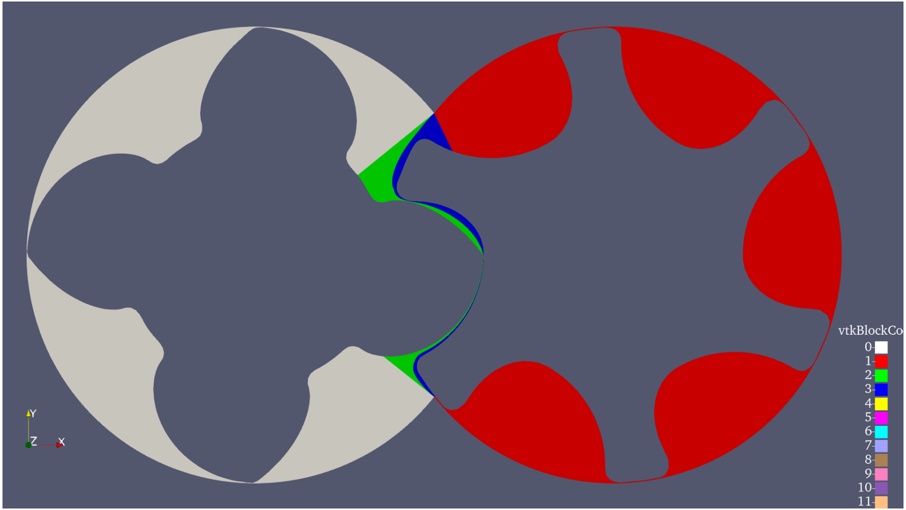


Application: twin-screw rotary compressor

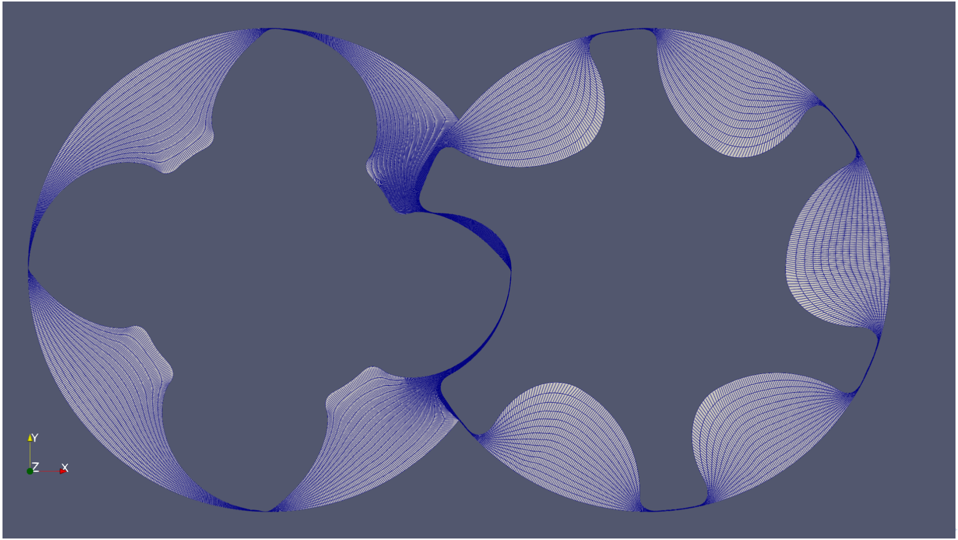




Application: twin-screw rotary compressor

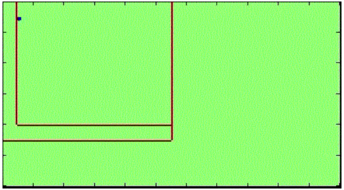


Application: twin-screw rotary compressor

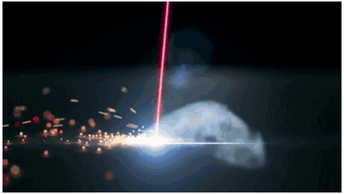


Catalogue

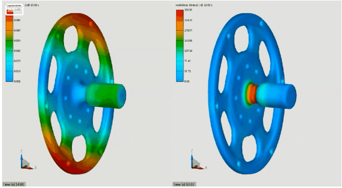
Anisotropic phenomena in physics



Wave propagation. source



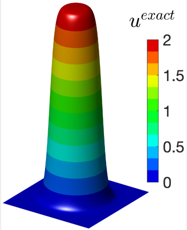
Laser printing. source



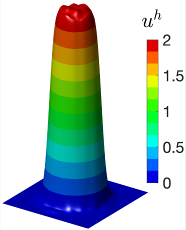
Stress concentration. source

- **Localized and anisotropic features extensively exist** in various physical phenomena;
- For such problems, **isotropic parameterizations are computationally uneconomical**;
- **Anisotropic parameterizations (r-adaptivity)**: increase per-degree-of-freedom accuracy while keeping the total degrees-of-freedom (DOFs) constant.

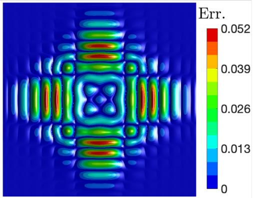
Basic Idea



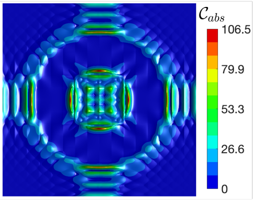
Exact solution



IGA solution



Absolute error



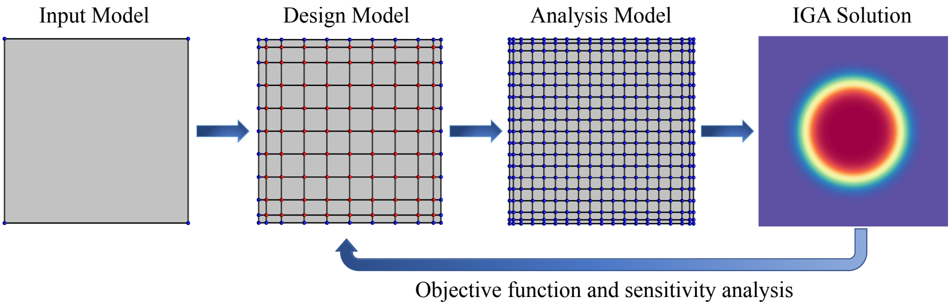
Absolute principal curvature

- **Absolute principal curvature**: to characterize the variations of the isogeometric solution;
- A tight relationship between geometric quantity and isogeometric solution is established;
- Absolute error and absolute principal curvature **show similar performance** (left figure);
- Absolute principal curvature is a good error estimator.

Basic idea - cont'd

Anisotropic parameterizations are often solution-dependent:

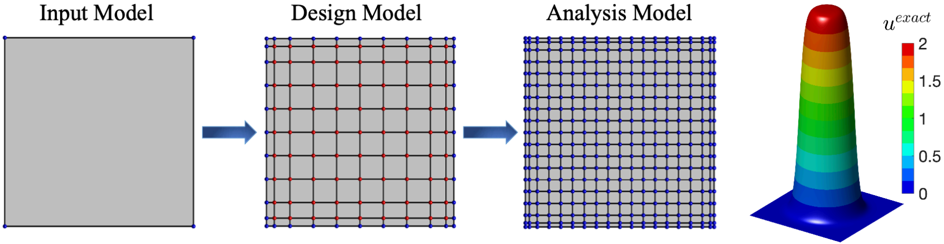
- Need **good numerical solution accuracy** to drive parameterization;
- Adjust as few control points as possible **for high efficiency**;
- **Bi-level strategy**: a coarse level (design model) to update the parameterization for efficiency's sake and a fine level (analysis model) to perform the isogeometric simulation for accuracy's sake.



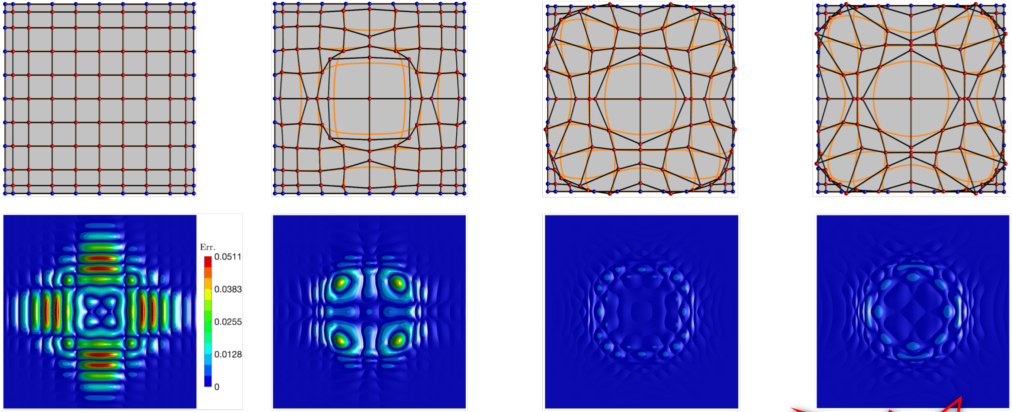
Toy problem: the square case

- Consider a Poisson's problem over $\Omega = [0,1]^2$ with the following exact solution

$$u(\boldsymbol{x}) = \tanh\left(\frac{0.25 - \sqrt{(x-0.5)^2 + (y-0.5)^2}}{0.05}\right) + 1.$$



Toy problem: the square case - cont'd



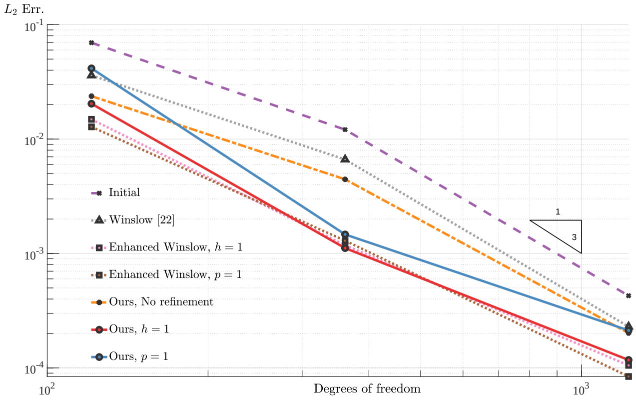
Initial (isotropic)
Err. = 1.2093e-02

No refinment
Err. = 4.4468e-03

$p = 1$
Err. = 1.4674e-03

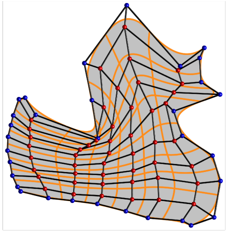
$h = 1$
Err. = 1.1141e-03

Toy problem: the square case - cont'd

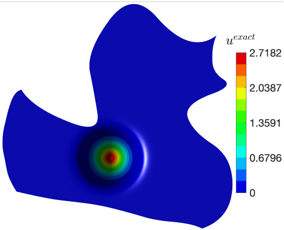


- High numerical accuracy **remains in finer meshes**;
- **Our method shows better performance** than [Xu et al. 2019].

More complicated geometry



Initial parameterization [Ji et al. 2021]



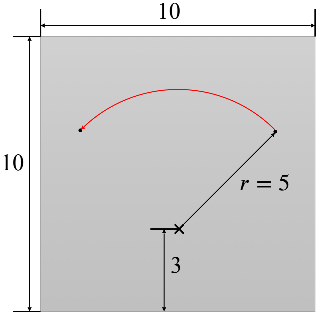
Exact solution

Refinement	Average error	L_2 error	Running time (sec.)
Initialization	1.5032e−02	2.7269e−02	0.21
NO	6.2290e−03	1.1213e−02	6.03
$p = 1$	**2.1423e−03**	**4.5594e−03**	9.28
$h = 1$	2.4356e−03	7.0504e−03	15.60

Application: Time-dependent dynamic PDE

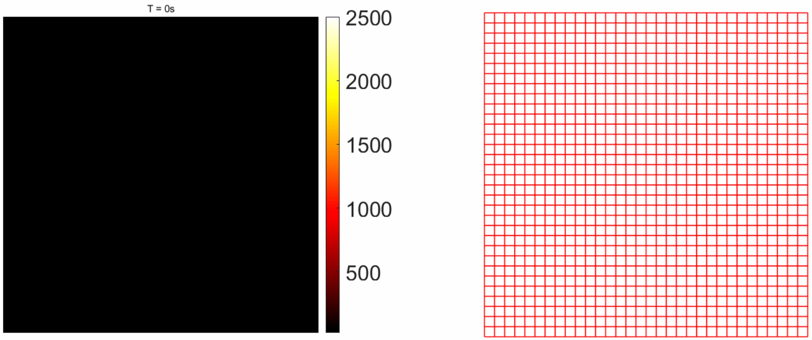
- Consider a 2D **linear heat transfer problem** with a moving Gaussian heat source:

$$\begin{cases} C_p \rho \frac{\partial u(\boldsymbol{x},t)}{\partial t} - \nabla \cdot (\kappa \nabla u(\boldsymbol{u}, t)) & \text{in } \Omega \times T \\ u(\boldsymbol{x}, t) = u_0 & \text{in } \Omega \\ \kappa \nabla u(\boldsymbol{x}, t) = 0 & \text{on } \partial\Omega \times T \end{cases}$$



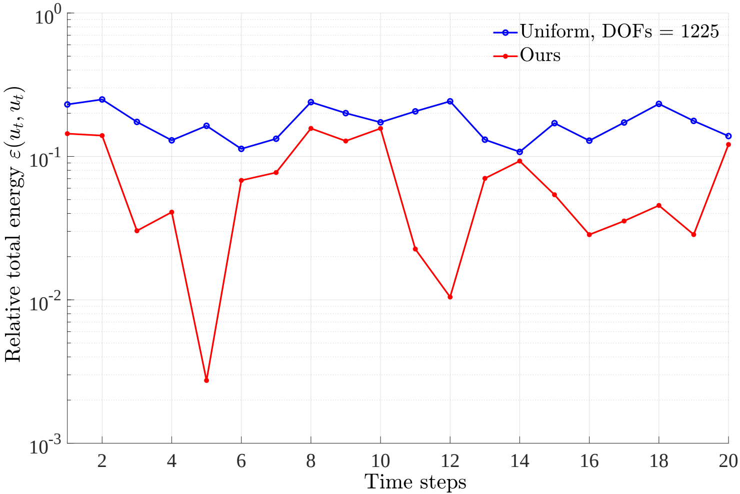
Laser Power P	9×10^5	$[W]$
Laser speed	1.57	$[mm/s]$
Absorptivity η	0.33	
Source radius r_h	100	$[\mu m]$
Conductivity κ	1.0	$[\mathrm{W}/mm/K]$
Heat capacity C_p	1.0	$[\mathrm{J/kg}/K]$
Density ρ	1.0	$[kg/mm^3]$
Initial temperature u_0	20.0	[°C]

Application: Time-dependent dynamic PDE



$u(\boldsymbol{x}, t)$ and their corresponding parameterizations on different time instants t

Application: Time-dependent dynamic PDE



Errors vs. Different time instants t

Catalogue

Conclusions and future work

- Conclusions:
  - **Barrier function-based NURBS parameterization method** is proposed for both the planar and volumetric cases;
  - **Penalty function-based NURBS parameterization method** is proposed for both the planar and volumetric cases;
  - **Full analytical gradient** is deduced to enhance the efficiency and robustness;
  - **Both of the proposed parameterization approaches work for the multi-patch cases**;
  - **Curvature based r-adaptive parameterization approach using bi-level strategy** is proposed to gain better numerical performance.

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- Future work:
  - **Role of the inner weights** on analysis-suitable parameterization construction;
  - Extend our parameterization method to **high genus computational domains**;

Conclusions and future work

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- Future work:
  - **Role of the inner weights** on analysis-suitable parameterization construction;
  - Extend our parameterization method to **high genus computational domains**;
  - In addition, we will **release all of the models and our reference implementation** in Geometry + Simulation Modules (**G+Smo**) library.

Thanks for your attention!

Q&A.

jiye@mail.dlut.edu.cn

Analysis-suitable Parameterization Construction and Curvature-based r-Adaptive Parameterization for IsoGeometric Analysis

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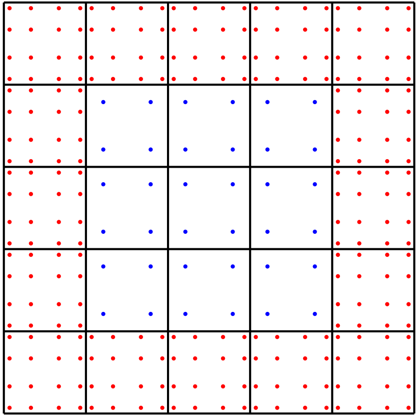
School of Mathematical Sciences, Dalian University of Technology, Dalian, China
Delft Institute of Applied Mathematics, Delft University of Technology, the Netherlands

Oct. 25, 2022
Nanjing University of Science and Technology

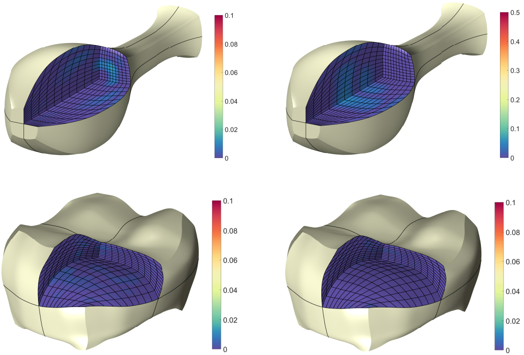
Reduced numerical integration scheme



Bi-cubic NURBS parameterization: 4×4 Gaussian integration points for the layer elements and 2×2 points for the inner elements.

- **OBSERVATION**: the Jacobians vary greatly near the boundary, but are often relatively flat inside.
- More integration points for the layer elements, and fewer integration points for the inner elements.
- In addition, we precompute the basis functions before iteration to further improve the computational efficiency.

Comparison: Reduced numerical integration vs. high precision integration



$\left| m_{SJ}^{\mathrm{HP}} - m_{SJ}^{\mathrm{Algo.}\ 1} \right|$ $\left| m_{unif.}^{\mathrm{HP}} - m_{unif.}^{\mathrm{Algo.}\ 1} \right|$

- Reduced integration strategy is adopted to accelerate the proposed method. However, *will this cause a loss of parameterization quality?*
- **NO!** The absolute differences of quality metrics are extremely close to 0.

Comparison: Reduced numerical integration vs. high precision integration

- However, it dramatically reduces the computational costs.