Curvature-based r-adaptive isogeometric analysis with injectivity-preserving multi-sided domain parameterization

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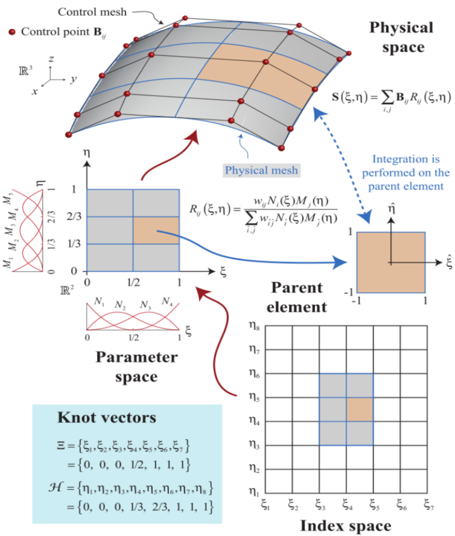
Catalogue



taken from [Cottrell+ 2009]

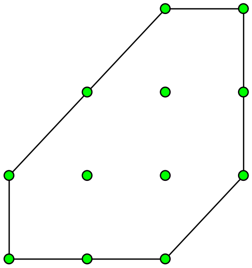
- Proposed by T.J.R. Hughes et al., 2005.
- KEY IDEA: approximate the physical fields with the same basis functions as that used to generate the CAD model.
- Advantages:
  - Integration of design and analysis
  - Exact and efficient CAD geometry
  - Flexible refinement scheme
  - High order continuous field
  - Superior approximation properties
- Very broad applications: such as shell analysis, fluid–structure interaction, and shape optimization.

Parameterization for IGA

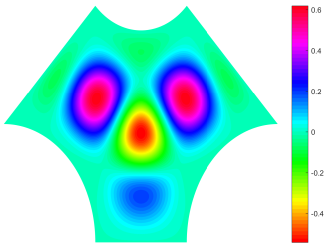
- Constructing analysis-suitable parameterization is a crucial step in IGA.
- Prerequisite: **injectivity**.
- Isotropic parameterization: orthogonality, uniformity and/or low distortion.
- **Anisotropic parameterization:**
  - r-adaptive method: redistribute and concentrate the isoparametric structure to some required regions while keeping their number and connectivity frozen.
  - required regions: with large gradient values and/or error.
  - more numerical accuracy and efficiency.
  - resulting parameterizations usually depend on the governing equation.

Multi-sided domain parameterization

- Multi-sided domain parameterization is emphasized.
- Toric surfaces (Krasauskas, 2002)
  - generalization of rational Bézier surfaces
  - congenital advantages in multi-sided computational domains modeling
  - successfully applied to IGA (Yu et al. 2020, Zhu et al. 2020, Li et al. 2021)



(a) lattice points



(b) toric IGA solution

Problem restatement

- Injectivity theorem: if compatibility, toric surface preserves injectivity for any set of positive weights (Craciun et al. 2008, Sottile and Zhu 2011).
- Thus weights are taken as optimization variables.
- Problem restatement: given the compatible lattice points set and control points set, the goal is to improve numerical accuracy by adjustment of the internal weights.

Related work

- Isotropic parameterizations: Cohen et al. 2010, Xu et al. 2011a 2011b 2013a 2013b 2015 2018 2019, Xu et al. 2014, Nian and Chen 2016, Lin et al. 2017 2019, Hinz et al. 2018, Ugalde et al. 2018, Pan et al. 2018 2020, Chen et al. 2019, Zheng et al. 2019, Liu et al. 2020, Yuan et al. 2021, Ji et al. 2021, etc.
- Anisotropic methods:
  - r-refinement method [Xu et al. 2011a]
  - analysis-oriented methods [Gravesen et al.2014]
  - efficient r-adaptive method [Xu et al. 2019]
  - PDE-based and solution-dependent parameterization [Ali and Ma 2021]
- Toric surface patches: Krasauskas 2002, Craciun et al. 2008, Sottile and Zhu 2011, Yu et al. 2020 2021, Zhu et al. 2020, Li et al. 2021.

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Toric surface patches

- lattice points set $\mathcal{A}$.
- parametric domain $\Delta_{\mathcal{A}} = CH(\mathcal{A})$.
- $L_i(\xi, \eta) = a_i\xi + b_i\eta + c_i = 0$.
- For each lattice point $\mathbf{a} \in \mathcal{A}$, we define toric-Bernstein basis function

$$\beta_{\mathbf{a}}(\xi, \eta) = c_{\mathbf{a}} \prod_{i=1}^{N} L_i(\xi, \eta)^{L_i(\mathbf{a})}. \qquad (1)$$

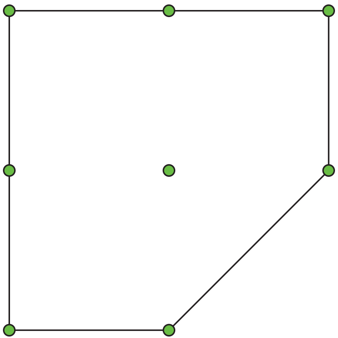


Fig. 1 lattice points

Toric surface patches - continued

- weights set $\boldsymbol{\omega} = \{\omega_{\mathbf{a}} | \omega_{\mathbf{a}} > 0, \mathbf{a} \in \mathcal{A}\}$
- rational toric-Bernstein basis functions:

$$T_{\mathbf{a}}(\xi, \eta) = \frac{\omega_{\mathbf{a}} \beta_{\mathbf{a}}(\xi, \eta)}{\Sigma_{\mathbf{a} \in \mathcal{A}} \omega_{\mathbf{a}} \beta_{\mathbf{a}}(\xi, \eta)}. \quad (2)$$

Definition 1

(Krasauskas 2002) Given a set of control points $\mathbf{P}$ and a set of positive weights $\boldsymbol{\omega}$ corresponding to a lattice points set $\mathcal{A}$, the toric surface patch is the image of the rational mapping $\mathcal{S} : \Delta_{\mathcal{A}} \to \mathbb{R}^d (d = 2 \text{ or } 3)$

$$\mathcal{S}(\xi, \eta) = \Sigma_{\mathbf{a} \in \mathcal{A}} \mathbf{P}_{\mathbf{a}} T_{\mathbf{a}}(\xi, \eta), (\xi, \eta) \in \Delta_{\mathcal{A}}. \quad (3)$$

Toric surface patches - continued

- multi-sided generalization of classical rational Bézier surfaces
- may degenerate into tensor product and triangular Bézier patches by proper selections of lattice points set $\mathcal{A}$ and coefficients $c_{\mathbf{a}}$
- nice geometric properties:
  - corner point interpolation
  - boundary property

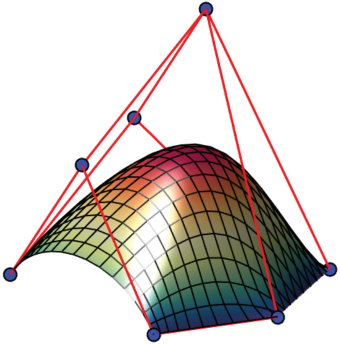


Fig. 2 toric surface

Injectivity condition

- injectivity is essential in isogeometric parameterization.
- Craciun et al. (2008) propose a necessary and sufficient condition for toric surface patches to guarantee injectivity for arbitrary a set of positive weights.
- However, a flaw is found and corrected by Sottile and Zhu (2011).
- We review the definition of compatibility first.

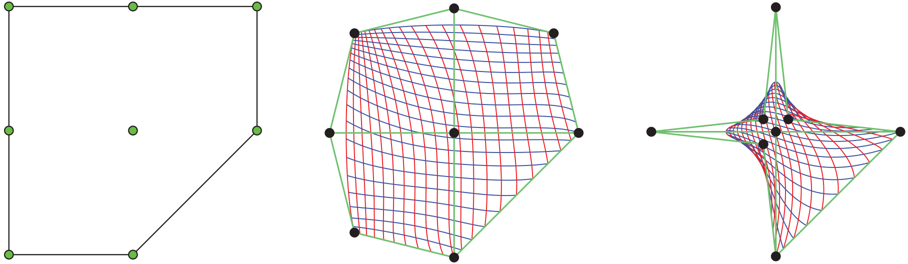
Definition 2

(Craciun et al. 2008, Sottile and Zhu 2011) The lattice points set $\mathcal{A}$ and the corresponding control points set $\mathbf{P}$ are called compatible if they satisfy the following three conditions:

1) There is at least one affinely independent triplet lattice points subset $\{\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k\} \subseteq \mathcal{A}$ such that the corresponding triplet control points subset $\{\mathbf{P}_{\mathbf{a}_i}, \mathbf{P}_{\mathbf{a}_j}, \mathbf{P}_{\mathbf{a}_k}\} \subseteq \mathbf{P}$ is also affinely independent,
2) For any two affinely independent triplet lattice points subsets that their corresponding triplet control points subsets are both affinely independent, if $\{\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k\}$ has the same orientation as $\{\tilde{\mathbf{a}}_i, \tilde{\mathbf{a}}_j, \tilde{\mathbf{a}}_k\}$, then $\{\mathbf{P}_{\mathbf{a}_i}, \mathbf{P}_{\mathbf{a}_j}, \mathbf{P}_{\mathbf{a}_k}\}$ has the same orientation as $\{\mathbf{P}_{\tilde{\mathbf{a}}_i}, \mathbf{P}_{\tilde{\mathbf{a}}_j}, \mathbf{P}_{\tilde{\mathbf{a}}_k}\}$, and
3) No two corner control points are consistent.

Theorem 3

(Craciun et al. 2008, Sottile and Zhu 2011) For arbitrary a set of positive weights, toric surface remains injectivity if and only if the lattice points set $\mathcal{A}$ and the control points set $\mathbf{P}$ are compatible.



(a) lattice points (b) compatiblity (c) incompatibility

Fig. 3 compatiblity and incompatibility with the same weights set

Catalogue

Model problem

- 2D Poisson's equation with homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u(\mathbf{x}) &= f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ u(\mathbf{x}) &= 0, \qquad \mathbf{x} \in \partial\Omega, \end{aligned} \tag{4}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ denotes the Laplacian operator.

IGA framework

- weak formulation

Model problem: Poisson's equation

find $u \in H^1(\Omega)$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} fv \, d\Omega,$$

for any $v \in H_0^1(\Omega)$.

IGA framework

- weak formulation
- parametric domain and parameterization

Model problem: Poisson's equation

toric parameterization:

$$\mathcal{S}(\xi,\eta)=\sum_{\mathbf{a}\in\mathcal{A}}T_{\mathbf{a}}(\boldsymbol{\xi})\mathbf{P}_{\mathbf{a}}=\sum_{i=1}^{\#\mathcal{A}}T_i(\boldsymbol{\xi})\mathbf{P}_i,\quad \boldsymbol{\xi}\in\Delta_{\mathcal{A}}.$$

IGA framework

- weak formulation
- parametric domain and parameterization
- discrete problem and spaces in the parametric and physical domain

Model problem: Poisson's equation

Galerkin formulation: find $u^h \in \mathcal{U}^h \subset H^1(\Omega)$ such that

$$\int_\Omega \nabla u^h \cdot \nabla v^h \, d\Omega = \int_\Omega f v^h \, d\Omega, \quad \forall v^h \in \mathcal{V}^h \subset H_0^1(\Omega),$$

where $\mathcal{U}^h$ and $\mathcal{V}^h$ are finite-dimensional subspace of $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively.

IGA framework

- weak formulation
- parametric domain and parameterization
- discrete problem and spaces in the parametric and physical domain
- construt and solve a linear system to find the discrete solution

Model problem: Poisson's equation

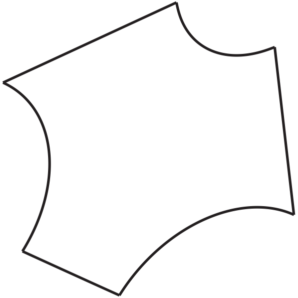
Trial function $u^h = \sum_{i=1}^{\#\mathcal{A}} T_i(\boldsymbol{\xi})u_i$ and test functions $T_j(\boldsymbol{\xi})$:

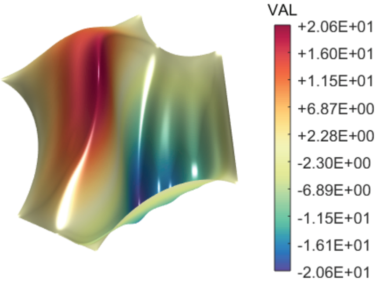
$$\sum_{i=1}^{\#\mathcal{A}} u_i \int_{\Delta_\mathcal{A}} (J^{-\mathrm{T}}\nabla_{\boldsymbol{\xi}} T_i)\cdot(J^{-\mathrm{T}}\nabla_{\boldsymbol{\xi}} T_j)|J|d\xi d\eta = \int_{\Delta_\mathcal{A}} f(\mathbf{x})T_j(\mathbf{x})|J|d\xi d\eta,\ j = 1, 2, \cdots, \#\mathcal{A}.$$

Toric IGA solution surfaces

- Numerical solution can be regarded as a parametric surface in $\mathbb{R}^3$, i.e.,

$$\mathcal{S}(\xi,\eta) = (x(\xi,\eta), y(\xi,\eta), u^h(\xi,\eta)). \tag{5}$$





- Curvatures is employed to characterize the gradient information of IGA solution.

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Absolute curvature metrics

- first and second fundamental forms of $\mathcal{S}(\xi, \eta)$ are defined by:

$$\mathrm{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} < \mathcal{S}_\xi, \mathcal{S}_\xi > & < \mathcal{S}_\xi, \mathcal{S}_\eta > \\ < \mathcal{S}_\xi, \mathcal{S}_\eta > & < \mathcal{S}_\eta, \mathcal{S}_\eta > \end{bmatrix}, \quad (6)$$

and

$$\mathrm{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} < \mathcal{S}_{\xi\xi}, \mathbf{n} > & < \mathcal{S}_{\xi\eta}, \mathbf{n} > \\ < \mathcal{S}_{\xi\eta}, \mathbf{n} > & < \mathcal{S}_{\eta\eta}, \mathbf{n} > \end{bmatrix}, \quad (7)$$

where $< \cdot, \cdot >$ denotes the inner product of two vectors and $\mathbf{n}$ is the unit normal vector.

- Gaussian curvature:

$$\mathcal{C}_K = \frac{det(\mathrm{II})}{det(\mathrm{I})} = \frac{LN - M^2}{EG - F^2}. \tag{8}$$

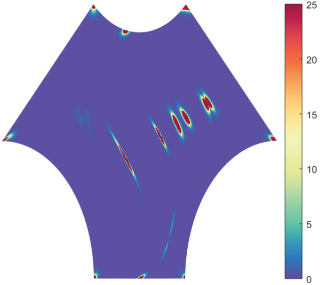
- mean curvature:

$$\mathcal{C}_H = \frac{LG - 2MF + NE}{2(EG - F^2)}. \tag{9}$$

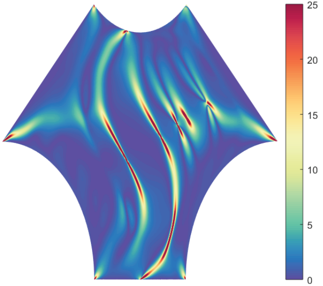
- principal curvatures:

$$\begin{aligned} k_{max} &= \mathcal{C}_H + \sqrt{\mathcal{C}_H^2 - \mathcal{C}_K} \\ k_{min} &= \mathcal{C}_H - \sqrt{\mathcal{C}_H^2 - \mathcal{C}_K} \end{aligned} \tag{10}$$

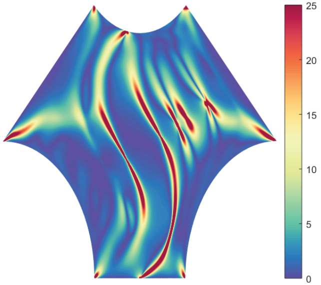
- absolute Gaussian curvature: $\mathcal{C}_{|K|} = |\mathcal{C}_K|$.
- absolute mean curvature: $\mathcal{C}_{|H|} = |\mathcal{C}_H|$.
- absolute principal curvature: $\mathcal{C}_{abs} = |k_{max}| + |k_{min}|$.



(a) absolute Gauss curvature



(b) absolute mean curvature



(c) absolute principal curvature

- absolute curvature metrics:

$$E(\omega;\mathcal{C}) = \int_{\Delta_{\mathcal{A}}} \mathcal{C}\ d\xi d\eta. \tag{11}$$

- curvature-based r-adaptive IGA method can be summarized as follows:

$$\begin{aligned} \arg\min_{\omega_i} \quad & E(\omega;\mathcal{C}) \\ s.t. \quad & \omega_i \geq \delta, \quad i = 1, 2, \cdots, \#\mathcal{A}, \end{aligned} \tag{12}$$

where $\delta > 0$ is a user-specified threshold.

Algorithm 1 Toric curvature-based r-adaptive method by weights adjustment.

Input: Initial weights ω^0, two compatible lattice points set $\mathcal{A}$ and control points set $\mathbf{P}$, convergence threshold ε and maximum iteration N_{max}.

Output: the optimal value of the inner weights.

1: Solve the governing PDE over given initial toric parameterization to obtain initial numerical solution and compute the curvature metrics in (11) of numerical solution surface;

2: Invoke an optimization solver to update the inner weights;

3: Solve the governing PDE with weights ω_{k+1} to obtain numerical solution surface in (5);

4: Compute curvature metrics $E(\omega^{k+1}; C)$ in (11);

5: If $|E(\omega^{k+1}; C) - E(\omega^k; C)|/|E(\omega^k; C)| < \varepsilon$ or $k > N_{max}$, terminate; Otherwise, set $\omega^k = \omega^{k+1}$ and return to Step 2.

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Numerical examples and comparisons

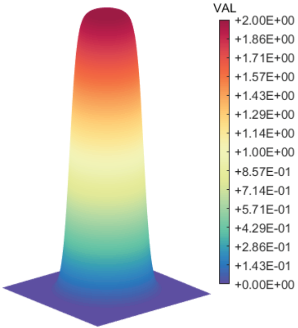
Implementation details

- MATLAB R2020b
- Optimizer: derivative-free BOBYQA in NLopt [1].
- Gaussian quadrature rules for integral items and the MATLAB backslash divide command for solving linear systems.
- All parameters involved in our algorithm are set as default values.
- The maximum number N_{max} of iterations and relative function value tolerance ε in Algorithm 1 are set as 1000 and $1e-6$, respectively. And the lower bound δ of weights in Problem (12) is set as $1e-3$.

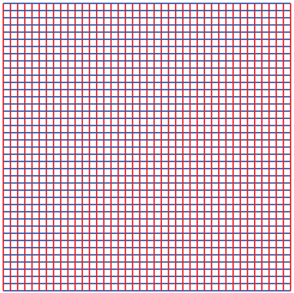
[1] S.G. Johnson, The NLopt nonlinear-optimization package, http://github.com/stevengj/nlopt.

Example I: A rectangular toric surface - classical rational Bézier case

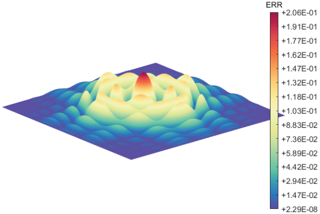
- computational domain: $\Omega = [0,1] \times [0,1]$
- exact solution: $T(\mathbf{x}) = tanh(\frac{0.25 - \sqrt{(x-0.5)^2+(y-0.5)^2}}{0.05}) + 1$



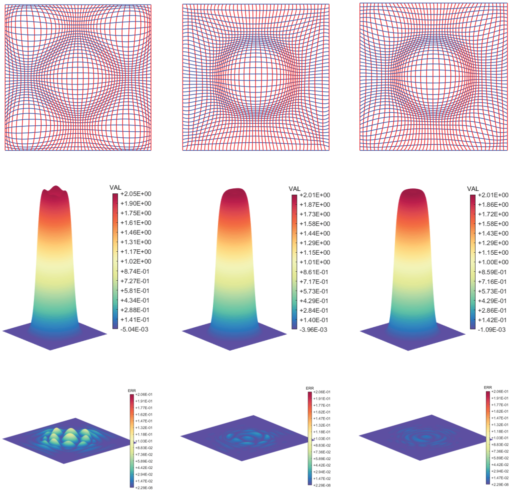
(a) exact solution



(b) initial parameterization



(c) initial error colormap



(a) $E(\omega; \mathcal{C}_{|K|})$ (b) $E(\omega; \mathcal{C}_{|H|})$ (c) $E(\omega; \mathcal{C}_{abs})$

- isoparametric lines concentrate on regions with large change of solution.
- $E(\omega; \mathcal{C}_{abs}) > E(\omega; \mathcal{C}_{|H|}) \gg E(\omega; \mathcal{C}_{|K|})$.

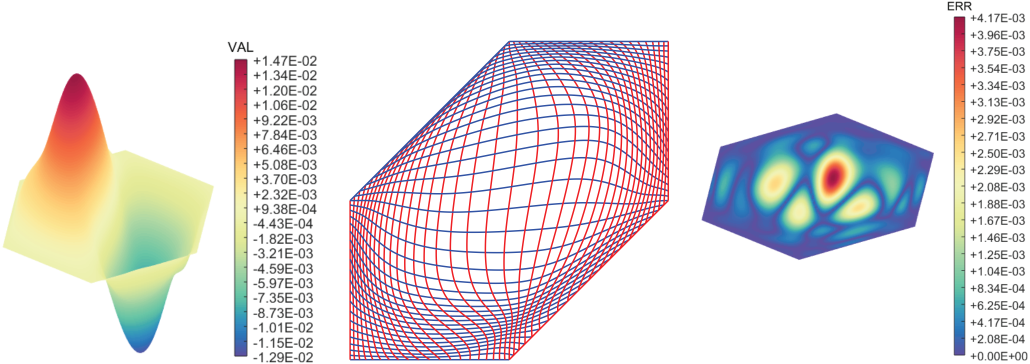
Example II: A hexagonal toric surface case

- consider a hexagonal computational domain $\Omega = \{(x,y) \in \mathbb{R}^2 | \ell_i(x,y) \geq 0,\ i = 1,2,\cdots,6\}$, where

$$\begin{aligned} \ell_1(x,y) &= y, & \ell_2(x,y) &= y - x + 1/2, \\ \ell_3(x,y) &= 1 - x, & \ell_4(x,y) &= 1 - y, \\ \ell_5(x,y) &= x - y + 1/2, & \ell_6(x,y) &= x \end{aligned} \tag{13}$$

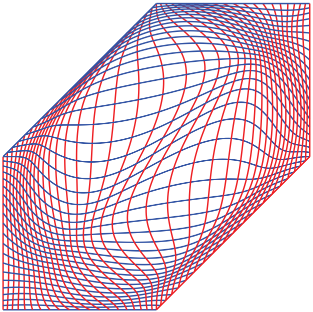
  parameterized by a hexagonal toric surface patch.
- exact solution: $T(\mathbf{x}) = \ell_1\ell_2\ell_3\ell_4\ell_5\ell_6 tanh(y - x)$.

Example II: A hexagonal toric surface case

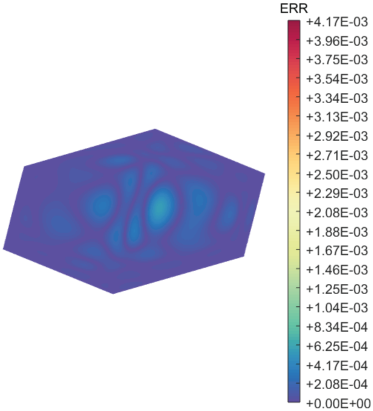


(a) exact solution (b) initial parameterization (c) initial error colormap

Example II: A hexagonal toric surface case

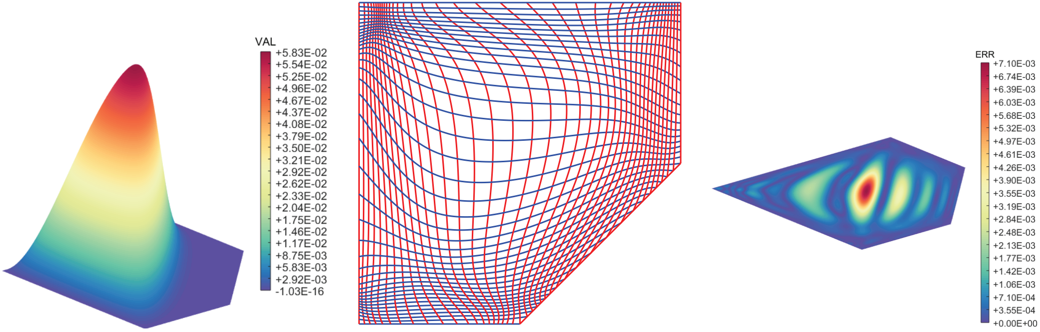


(a) optimal parameterization with $E(\omega;\mathcal{C}_{abs})$



(b) error colormap with $E(\omega;\mathcal{C}_{abs})$

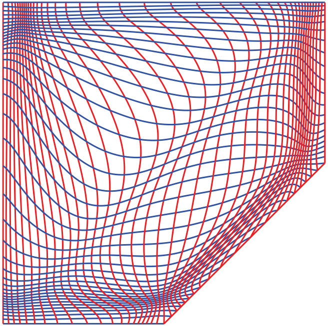
Example III: A pentagonal toric surface case



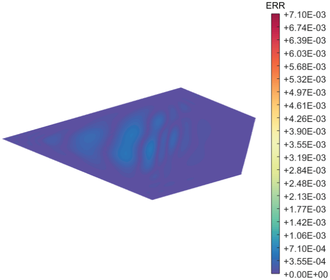
(a) exact solution (b) initial parameterization (c) initial error colormap

Fig. 4 Example III: the exact solution, initial parameterization and initial error colormap.

Example III: A pentagonal toric surface case



(a) optimal parameterization with $E(\omega; \mathcal{C}_{abs})$



(b) error colormap with $E(\omega; \mathcal{C}_{abs})$

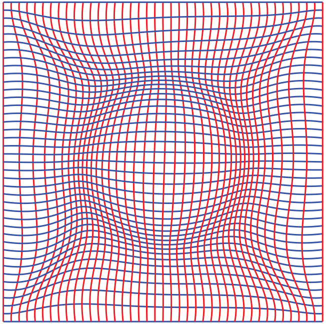
Comparisons with Xu et al. 2019

- Xu et al. minimize the following objective function:

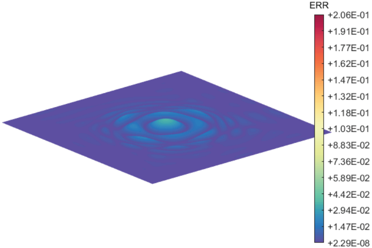
$$E_W = \frac{1}{2}\int_{\Delta_{\mathcal{A}}} \mathcal{C}_{abs}\left(\frac{x_\xi^2 + x_\eta^2 + y_\xi^2 + y_\eta^2}{det(J(\boldsymbol{\xi}))}\right) d\xi d\eta \tag{14}$$

  where $det(J(\boldsymbol{\xi}))$ is the Jacobian determinant of parameterization.
- inner control points are taken as variables.
- to preserve injectivity, Winslow's term is added.
- different from the original method, we directly minimize (14).

Comparisons with Xu et al. 2019



(a) optimal parameterization



(b) error colormap

Fig. 5 Example I: the optimal parameterization and the error colormap obtained by solving Winslow's method.

Table 1 Quantitative data for the examples with various methods.

Example	#DOF	Curvature metrics	Max error	Average error	L_2 error	running time (s)
Example I	13×13	Initialization	2.0610E-1	2.9938E-2	4.6741E-2	-
		$E(\omega; \mathcal{C}_{\lvert K\rvert})$	8.7268E-2	1.1625E-2	1.9447E-2	19.20
		$E(\omega; \mathcal{C}_{\lvert H\rvert})$	2.7523E-2	3.2311E-3	5.4661E-3	19.35
		$E(\omega; \mathcal{C}_{abs})$	**1.6132E-2**	**2.3163E-3**	**3.6399E-3**	**14.70**
		Xu et al. 2019	3.6061E-2	3.0458E-3	5.0327E-3	23.28
Example II	91	Initialization	4.1699E-3	2.5169E-4	4.2484E-4	-
		$E(\omega; \mathcal{C}_{\lvert K\rvert})$	5.9083E-4	9.6567E-5	1.2424E-4	4.92
		$E(\omega; \mathcal{C}_{\lvert H\rvert})$	5.9386E-4	4.1952E-5	6.1472E-5	4.96
		$E(\omega; \mathcal{C}_{abs})$	**5.7617E-4**	**4.0800E-5**	**5.7336E-5**	**2.88**
		Xu et al. 2019	1.2969E-3	8.5169E-5	1.3278E-4	6.83
Example III	148	Initialization	7.0986E-03	4.1455E-04	7.0983E-04	-
		$E(\omega; \mathcal{C}_{\lvert K\rvert})$	1.1926E-03	1.4319E-04	2.2224E-04	16.62
		$E(\omega; \mathcal{C}_{\lvert H\rvert})$	**4.7574E-04**	**3.1831E-05**	**5.9611E-05**	16.35
		$E(\omega; \mathcal{C}_{abs})$	5.1989E-04	3.8627E-05	7.2235E-05	**14.42**
		Xu et al. 2019	6.3170E-04	8.1240E-05	1.1314E-04	16.44

Catalogue

Conclusions and future work

Conclusions

- Curvature-based r-adaptive isogeometric method for 2D multi-sided computational domains is proposed.
- Three absolute curvature metrics of isogeometric solution surface to characterize its gradient information.
- Numerical examples demonstrate the effectiveness and efficiency of our method.

Future work

- Multi-patch techniques combining the continuity conditions of toric surface patches (Sun et al. 2015) could be a part of future work.
- Extend our approach to time-dependent dynamic PDEs and $3D$ volumetric problems.

Thanks for your attention!

Q&A.

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